# GENERALIZED DESCENT ALGEBRA AND CONSTRUCTION OF IRREDUCIBLE CHARACTERS OF HYPEROCTAHEDRAL GROUPS

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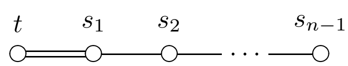
## 1. Introduction

Let $(W,S)$ be a finite Coxeter system and let $\ell : W \to \mathbb{N}$ denote the length function. If $I \subset S$, $W_I =< I >$ is the standard parabolic subgroup generated by $I$ and $X_I = \{w \in W \mid \forall\ s \in I,\ \ell(ws) > \ell(w)\}$ is a cross-section of $W/W_I$. Write $x_I = \sum_{w \in X_I} w \in \mathbb{Z}W$, then $\Sigma(W) = \oplus_{I \subset S} \mathbb{Z} x_I$ is a subalgebra of $\mathbb{Z}W$ and the $\mathbb{Z}$-linear map $\theta : \Sigma(W) \to \mathbb{Z}\operatorname{Irr} W$, $x_I \mapsto \operatorname{Ind}_{W_I}^{W} 1$ is a morphism of algebras: $\Sigma(W)$ is called the *descent algebra* or the *Solomon algebra* of $W$ [18]. However, the morphism $\theta$ is surjective if and only if $W$ is a product of symmetric groups.

The aim of this paper is to construct, whenever $W$ is of type $C$, a subalgebra $\Sigma'(W)$ of $\mathbb{Z}W$ containing $\Sigma(W)$ and a surjective morphism of algebras $\theta'$ : $\Sigma'(W) \to \mathbb{Z}\operatorname{Irr} W$ build similarly as $\Sigma(W)$ by starting with a bigger generating set. More precisely, let $(W_n, S_n)$ denote a Coxeter system of type $C_n$ and write $S_n = \{t, s_1, \dots, s_{n-1}\}$ where the Dynkin diagram of $(W_n, S_n)$ is




Let $t_1 = t$ and $t_i = s_{i-1} t_{i-1} s_{i-1}$ $(2 \le i \le n)$ and $S'_n = S_n \cup \{t_1, \dots, t_n\}$. Let $\mathcal{P}_0(S'_n)$ denote the set of subsets $I$ of $S'_n$ such that $I =< I > \cap S'_n$. If $I \in \mathcal{P}_0(S'_n)$, let $W_I$, $X_I$ and $x_I$ be defined as before. Then:

**Theorem.** *$\Sigma'(W_n) = \oplus_{I \in \mathcal{P}_0(S'_n)} \mathbb{Z} x_I$ is a subalgebra of $\mathbb{Z}W_n$ and the $\mathbb{Z}$-linear map $\theta_n : \Sigma'(W_n) \to \mathbb{Z}\operatorname{Irr} W_n$, $x_I \mapsto \operatorname{Ind}_{W_I}^{W} 1$ is a surjective morphism of algebras. Moreover, $\operatorname{Ker}\theta_n = \sum_{I \equiv I'} \mathbb{Z}(x_I - x_{I'})$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \operatorname{Ker}\theta_n$ is the radical of the $\mathbb{Q}$-algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \Sigma'(W_n)$.*

In this theorem, the notation $I \equiv I'$ means that there exists $w \in W_n$ such that $I' = {}^{w}I$, that is, $W_I$ and $W_{I'}$ are conjugated. This theorem is stated and proved in §3.3. Note that it is slightly differently formulated: in fact, it turns out that there is a natural bijection between signed compositions of $n$ and $\mathcal{P}_0(S'_n)$ (see Lemma 2.5). So, everything in the text is indexed by signed compositions instead of $\mathcal{P}_0(S'_n)$. It must also be noticed that, by opposition with the classical case, the multiplication $x_I x_J$ may involve negative coefficients. Using another basis, we show that $\Sigma'(W_n)$ is precisely the generalized descent algebra discovered by Mantaci and Reutenauer [16].

Using this theorem and the Robinson-Schensted correspondence for type $C$ constructed by Stanley [20] and a Knuth version of it given in [5], we obtain an analog

of Jöllenbeck's result (on the construction of characters of the symmetric group [12]) using an extension $\tilde{\theta}_n : \mathcal{Q}_n \to \mathbb{Z}\,\mathrm{Irr}\,W_n$ of $\theta_n$ to the coplactic space $\mathcal{Q}_n$ (see Theorem 4.14). The coplactic space refer to Jöllenbeck's construction revised in [3].

Now, let $\mathcal{SP} = \oplus_{n\geq 0}\mathbb{Z}W_n$, $\Sigma' = \oplus_{n\geq 0}\Sigma'(W_n)$ and $\mathcal{Q} = \oplus_{n\geq 0}\mathcal{Q}_n$. Let $\theta = \oplus_{n\geq 0}\theta_n$ and $\tilde{\theta} = \oplus_{n\geq 0}\tilde{\theta}_n$. Aguiar and Mahajan have proved that $\mathcal{SP}$ is naturally a Hopf algebra and that $\Sigma'$ is a Hopf subalgebra [1]. We prove here that $\mathcal{Q}$ is also a Hopf subalgebra of $\mathcal{SP}$ (containing $\Sigma'$) and that $\theta$ and $\tilde{\theta}$ are surjective morphisms of Hopf algebras (see Theorem 5.8). This generalizes similar results in symmetric groups ([17] and [3]), which are parts of combinatorial tools used within the framework of the representation theory of type $A$ (see for instance [21]).

In the last section of this paper, we give some explicit computations in $\Sigma'(W_2)$ (characters, complete set of orthogonal primitive idempotents, Cartan matrix of $\Sigma'(W_2)$...).

In the Appendix, P. Baumann and the second author link the above construction with the Specht construction and symmetric functions (see [14]).

*Remark.* It seems interesting to try to construct a subalgebra $\Sigma'(W)$ of $\mathbb{Z}W$ containing $\Sigma(W)$ and a morphism $\theta' : \Sigma'(W) \to \mathbb{Z}\,\mathrm{Irr}\,W$ for arbitrary Coxeter group $W$. But it is impossible to do so in a same way as we did for type $C$ (by extending the generating set). Computations using `CHEVIE` programs show us that it is impossible to do so in type $D_4$ and that the reasonable choices in $F_4$ fail (we do not obtain a subalgebra!). However, it is possible to do something similar for type $G_2$. More precisely, let $(W,S)$ be of type $G_2$. Write $S = \{s,t\}$ and let $S' = \{s,t,sts,tstst\}$ and repeat the procedure described above to obtain a sub-$\mathbb{Z}$-module $\Sigma'(W)$ of $\mathbb{Z}W$ and a morphism $\theta' : \Sigma'(W) \to \mathbb{Z}\,\mathrm{Irr}\,W$. Then the theorem stated in this introduction also holds in this case. We have $\mathrm{rank}_{\mathbb{Z}}\,\Sigma'(W) = 8$ and $\mathrm{rank}_{\mathbb{Z}}\,\mathrm{Ker}\,\theta' = 2$.

## 2. Some reflection subgroups of hyperoctahedral groups

In this article, we denote $[m,n] = \{i \in \mathbb{Z} \mid m \leq i \leq n\} = \{m, m+1, \dots, n-1, n\}$, for all $m \leq n \in \mathbb{Z}$, and $\mathrm{sign}\,(i) \in \{\pm 1\}$ the sign of $i \in \mathbb{Z} \setminus \{0\}$. If $E$ is a set, we denote by $\mathfrak{S}(E)$ the group of permutations on the set $E$. If $m \in \mathbb{Z}$, we often denote by $\overline{m}$ the integer $-m$.

2.1. **The hyperoctahedral group.** We begin by making clear some notations and definitions concerning the *hyperoctahedral group* $W_n$. Denote $1_n$ the identity of $W_n$ (or 1 if no confusion is possible). We denote by $\ell_t(w)$ the number of occurrences of $t$ in a reduced decomposition of $w$ and we define $\ell_s(w) = \ell(w) - \ell_t(w)$.

It is well-known that $W_n$ acts on the set $I_n = [1,n] \cup [\bar{n}, \bar{1}]$ by permutations as follows: $t = (\bar{1},\ 1)$ and $s_i = (\overline{i+1}, \bar{i})(i, i+1)$ for any $i \in [1, n-1]$. Through this action, we have

$$W_n = \{w \in \mathfrak{S}(I_n) \mid \forall\ i \in I_n,\ w(\,\overline{i}\,) = \overline{w(i)}\}.$$

We often represent $w \in W_n$ as the word $w(1)w(2)\dots w(n)$ in examples.

The subgroup $W_{\bar{n}} = \{w \in W_n \mid w([1,n]) = [1,n]\}$ of $W_n$ is naturally identified with $\mathfrak{S}_n$, the symmetric group of degree $n$, by restriction of its elements to $[1,n]$. Note that $W_{\bar{n}}$ is generated, as a reflection subgroup of $W_n$, by $S_{\bar{n}} = \{s_1, \dots, s_{n-1}\}$.

A *standard parabolic subgroup* of $W_n$ is a subgroup generated by a subset of $S_n$ (a *parabolic subgroup* of $W_n$ is a subgroup conjugate to some standard parabolic subgroup). Note that $(W_{\bar{n}}, S_{\bar{n}})$ is a Coxeter group, which is a standard parabolic subgroup of $W_n$. If $m \leq n$, then $S_m$ is naturally identified with a subset of $S_n$ and $W_m$ will be identified with the standard parabolic subgroup of $W_n$ generated by $S_m$.

Now, we set $T_n = \{t_1, \dots, t_n\}$, with $t_i$ as in Introduction. As a permutation of $I_n$, note that $t_i = (i, \bar{i})$, then the reflection subgroup $\mathfrak{T}_n$ generated by $T_n$ is naturally identified with $(\mathbb{Z}/2\mathbb{Z})^n$. Therefore $W_n = W_{\bar{n}} \ltimes \mathfrak{T}_n$ is just the wreath product of $\mathfrak{S}_n$ by $\mathbb{Z}/2\mathbb{Z}$. If $w \in W_n$, we denote by $(w_S, w_T)$ the unique pair in $\mathfrak{S}_n \times \mathfrak{T}_n$ such that $w = w_S w_T$. Note that $\ell_t(w) = \ell_t(w_T)$. In this article, we will consider reflection subgroups generated by subsets of $S'_n = S_n \cup T_n$.

2.2. **Root system.** Before studying the reflection subgroups generated by subsets of $S'_n$, let us recall some basic facts about Weyl groups of type $C$ (see [6]). Let us endow $\mathbb{R}^n$ with its canonical euclidean structure. Let $(e_1, \dots, e_n)$ denote the canonical basis of $\mathbb{R}^n$: this is an orthonormal basis. If $\alpha \in \mathbb{R}^n \setminus \{0\}$, we denote by $s_\alpha$ the orthogonal reflection such that $s_\alpha(\alpha) = -\alpha$. Let

$$\Phi_n^+ = \{2e_i \mid 1 \leq i \leq n\} \cup \{e_j + \nu e_i \mid \nu \in \{1, -1\} \text{ and } 1 \leq i < j \leq n\},$$

$\Phi_n^- = -\Phi_n^+$ and $\Phi_n = \Phi_n^+ \cup \Phi_n^-$. Then $\Phi_n$ is a root system of type $C_n$ and $\Phi_n^+$ is a positive root system of $\Phi_n$. By sending $t$ to $s_{2e_1}$ and $s_i$ to $s_{e_{i+1}-e_i}$ (for $1 \leq i \leq n-1$), we will identify $W_n$ with the Coxeter group of $\Phi_n$. Then

$$\Delta_n = \{2e_1, e_2 - e_1, e_3 - e_2, \dots, e_n - e_{n-1}\}$$

is the basis of $\Phi_n$ contained in $\Phi_n^+$ and the subset $S_n$ of $W_n$ is naturally identified with the set of simple reflections $\{s_\alpha \mid \alpha \in \Delta_n\}$. Therefore, for any $w \in W_n$ we have

$$\ell(w) = |\Phi_n^+ \cap w^{-1}(\Phi_n^-)|;$$

and $\ell(ws_\alpha) < \ell(w)$ if and only if $w(\alpha) \in \Phi^-$, for all $\alpha \in \Phi^+$.

*Remark 2.1* - Let $w \in W_n$ and let $\alpha \in \Phi_n^+$. Then $\ell(ws_\alpha) < \ell(w)$ if and only if $w(\alpha) \in \Phi_n^-$. Therefore, if $i \in [1, n-1]$, then

$$\ell(ws_i) < \ell(w) \Leftrightarrow w(i) > w(i+1),$$

and, if $j \in [1, n]$, then

$$\ell(wt_j) < \ell(w) \Leftrightarrow w(j) < 0.$$

Therefore, we deduce from the strong exchange condition (see [11, §5.8])

$$\ell_t(w) = |\{i \in [1, n] \mid w(i) < 0\}|. \tag{2.2}$$

2.3. **Some closed subsystems of $\Phi_n$.** Consider the subsets $\{s_1, t_1\}$ and $\{s_1, t_2\}$ of $S'_n$ ($n \geq 2$). It is readily seen that these two sets of reflections generate the same reflection subgroup of $W_n$. This lead us to find a parametrization of subgroups generated by a subset of $S'_n$.

A *signed composition* is a sequence $C = (c_1, \dots, c_r)$ of non-zero elements of $\mathbb{Z}$. The number $r$ is called the *length* of $C$. We set $|C| = \sum_{i=1}^r |c_i|$. If $|C| = n$, we say that $C$ is a *signed composition of* $n$ and we write $C \mathrel{||}\!\!\models n$. We also define $C^+ = (|c_1|, \dots, |c_r|) \mathrel{||}\!\!\models n$, $C^- = -C^+$ and $\overline{C} = -C$. We denote by $\mathrm{Comp}(n)$ the set

of signed compositions of $n$. In particular, any composition is a signed composition (any part is positive). Note that

**(2.3)** $$|\operatorname{Comp}(n)| = 2.3^{n-1}.$$

Now, to each $C = (c_1, \dots, c_r) \mid\!\!\models n$, we associate a reflection subgroup of $W_n$ as follows: for $1 \le i \le r$, set

$$I_C^{(i)} = \begin{cases} I_{C,+}^{(i)} & \text{if } c_i < 0, \\ I_{C,+}^{(i)} \cup -I_{C,+}^{(i)} & \text{if } c_i > 0, \end{cases}$$

where $I_{C,+}^{(i)} = \big[|c_1| + \cdots + |c_{i-1}| + 1, |c_1| + \cdots + |c_i|\big]$. Then

$$W_C = \{w \in W_n \mid \forall\, 1 \le i \le r,\ w(I_C^{(i)}) = I_C^{(i)}\}$$

is a reflection subgroup generated by

$$\begin{aligned} S_C \quad = \quad & \{s_p \in S_{\bar{n}} \mid |c_1| + \cdots + |c_{i-1}| + 1 \le p \le |c_1| + \cdots + |c_i| - 1\} \\ & \cup \big\{t_{|c_1|+\cdots+|c_{j-1}|+1} \in T_n \mid c_j > 0\big\} \quad \subset S'_n \end{aligned}$$

Therefore, $W_C \simeq W_{c_1} \times \cdots \times W_{c_r}$: we denote by $(w_1, \dots, w_r) \mapsto w_1 \times \cdots \times w_r$ the natural isomorphism $W_{c_1} \times \cdots \times W_{c_r} \xrightarrow{\sim} W_C$.

*Example.* The group $W_{(\overline{2},3,\overline{1},\overline{3},1)} \simeq \mathfrak{S}_2 \times W_3 \times \mathfrak{S}_1 \times \mathfrak{S}_3 \times W_1$ is generated, as a reflection subgroup of $W_{10}$, by $S_{(\overline{2},3,\overline{1},\overline{3},1)} = \{s_1\} \cup \{t_3, s_3, s_4\} \cup \{s_7, s_8\} \cup \{t_{10}\} \subset S'_{10}$.

The signed composition $C$ is said *semi-positive* if $c_i \ge -1$ for every $i \in [1, r]$. Note that a composition is a semi-positive composition. We say that $C$ is *negative* if $c_i < 0$ for every $i \in [1, r]$. We say that $C$ is *parabolic* if $c_i < 0$ for $i \in [2, r]$. Note that $C$ is parabolic if and only if $W_C$ is a standard parabolic subgroup.

Now, let $S'_C = S'_n \cap W_C$, $\Phi_C = \{\alpha \in \Phi_n \mid s_\alpha \in W_C\}$ and $\Phi_C^+ = \Phi_C \cap \Phi_n^+$. Then $W_C$ is the Weyl group of the closed subsystem $\Phi_C$ of $\Phi_n$. Moreover, $\Phi_C^+$ is a positive root system of $\Phi_C$ and we denote by $\Delta_C$ the basis of $\Phi_C$ contained in $\Phi_C^+$. Note that $S_C = \{s_\alpha \mid \alpha \in \Delta_C\}$, so $(W_C, S_C)$ is a Coxeter group.

Let $\ell_C : W_C \to \mathbb{N}$ denote the length function on $W_C$ with respect to $S_C$. Let $w_C$ denote the longest element of $W_C$ with respect to $\ell_C$. If $C$ is a composition, we denote by $\sigma_C$ the longest element of $\mathfrak{S}_C = W_{\overline{C}}$ with respect to $\ell_{\overline{C}}$ (which is the restriction of $\ell$ to $\mathfrak{S}_C$). In other words, $\sigma_C = w_{\overline{C}}$. In particular, $w_n$ (resp. $\sigma_n$) denotes the longest element of $W_n$ (resp. $\mathfrak{S}_n$).

Write $T_C = T_n \cap W_C$ and $\mathfrak{T}_C = \mathfrak{T}_n \cap W_C$, then observe that

**(2.4)** $$W_C = W_{C^-} \ltimes \mathfrak{T}_C = \mathfrak{S}_{C^+} \ltimes \mathfrak{T}_C.$$

*Remarks.* (1) This class of reflection subgroups contains the standard parabolic subgroups, since $S_n \subset S'_n$. But it contains also some other subgroups which are not parabolic (consider the subgroup generated by $\{t_1, t_2\}$ as example). In other words, it may happen that $\Delta_C \not\subset \Delta_n$. In fact, $\Delta_C \subset \Delta_n$ if and only if $W_C$ is a standard parabolic subgroup of $W_n$.

(2) If $W_C$ is not a standard parabolic subgroup of $W_n$, then $\ell_C$ is not the restriction of $\ell$ to $W_C$.

We close this subsection by an easy characterization of the subsets $S'_C$:

**Lemma 2.5.** *Let $X$ be a subset of $S'_n$. Then the following are equivalent:*

(1) $< X > \cap S'_n = X$.
(2) $X \cap T_n$ *is stable under conjugation by* $< X >$.
(3) $X \cap T_n$ *is stable under conjugation by* $< X \cap S_{\bar{n}} >$.
(4) *There exists a signed composition* $C$ *of* $n$ *such that* $X = S'_C$.

**Corollary 2.6.** *Let* $w \in W_n$ *and let* $C \Vdash n$. *If* ${}^wS'_C \subset S'_n$, *then there exists a (unique) signed composition* $D$ *such that* ${}^wS'_C = S'_D$.

*Proof.* Indeed, ${}^wS'_C \cap T_n = {}^w(S'_C \cap T_n)$ and ${}^wS'_C \cap S_{\bar{n}} = {}^w(S'_C \cap S_{\bar{n}})$. □

2.4. **Orbits of closed subsystems of $\Phi_n$.** In this subsection, we determine when two subgroups $W_C$ and $W_D$ of $W_n$ are conjugated. A *bipartition* of $n$ is a pair $\lambda = (\lambda^+, \lambda^-)$ of partitions such that $|\lambda| := |\lambda^+| + |\lambda^-| = n$. We write $\lambda \Vdash n$ to say that $\lambda$ is a bipartition of $n$, and the set of bipartitions of $n$ is denoted by $\mathrm{Bip}(n)$. It is well-known that the conjugacy classes of $W_n$ are in bijection with $\mathrm{Bip}(n)$ (see [9, 14]). We define $\hat{\lambda}$ as the signed composition of $n$ obtained by concatenation of $\lambda^+$ and $-\lambda^-$. The map $\mathrm{Bip}(n) \to \mathrm{Comp}(n)$, $\lambda \mapsto \hat{\lambda}$ is injective.

Now, let $C$ be a signed composition of $n$. We define $\boldsymbol{\lambda}(C) = (\lambda^+, \lambda^-)$ as the bipartition of $n$ such that $\lambda^+$ (resp. $\lambda^-$) is obtained from $C$ by reordering in decreasing order the positive parts of $C$ (resp. the absolute value of the negative parts of $C$). One can easily check that the map

$$\boldsymbol{\lambda} : \mathrm{Comp}(n) \longrightarrow \mathrm{Bip}(n)$$

is surjective (indeed, if $\lambda \in \mathrm{Bip}(n)$, then $\boldsymbol{\lambda}(\hat{\lambda}) = \lambda$) and that the following proposition holds:

**Proposition 2.7.** *Let* $C, D \Vdash n$, *then* $W_C$ *and* $W_D$ *are conjugate in* $W_n$ *if and only if* $\boldsymbol{\lambda}(C) = \boldsymbol{\lambda}(D)$. *If* $\Psi$ *is a closed subsystem of* $\Phi_n$, *then there exists a unique bipartition* $\lambda$ *of* $n$ *and some* $w \in W_n$ *such that* $\Psi = w(\Phi_{\hat{\lambda}})$.

Let $C, D \Vdash n$, then we write $C \subset D$ if $W_C \subset W_D$. Moreover, $C$, $C' \subset D$ and if $W_C$ and $W_{C'}$ are conjugate under $W_D$, then we write $C \equiv_D C'$.

2.5. **Distinguished coset representatives.** Let $C \Vdash n$, then

$$X_C = \{x \in W_n \mid \forall\ w \in W_C,\ \ell(xw) \geq \ell(x)\}$$

is a distinguished set of *minimal coset representatives* for $W_n/W_C$ (see proposition below). It is readily seen that

$$\begin{aligned} X_C &= \{w \in W_n \mid w(\Phi_C^+) \subset \Phi_n^+\} \\ &= \{w \in W_n \mid \forall\ \alpha \in \Delta_C,\ w(\alpha) \in \Phi_n^+\}. \end{aligned}$$

Finally

$$X_C = \{w \in W_n \mid \forall\ r \in S_C,\ \ell(wr) > \ell(w)\}.$$

We need a relative notion: if $D \Vdash n$ such that $C \subset D$, the set $X_C^D = X_C \cap W_D$ is a distinguished set of minimal coset representatives for $W_D/W_C$. If $D = (n)$ we write $X_C^n$ instead of $X_C^{(n)}$.

**Proposition 2.8.** *Let* $C \Vdash n$, *then:*

(a) *The map* $X_C \times W_C \to W_n$, $(x, w) \mapsto xw$ *is bijective.*
(b) *If* $C \subset D$, *then the map* $X_D \times X_C^D \to X_C$, $(x, y) \mapsto xy$ *is bijective.*
(c) *If* $x \in X_C$ *and* $w \in W_C$, *then* $\ell_t(xwx^{-1}) \geq \ell_t(w)$. *Consequently,* $\mathfrak{S}_n \cap {}^xW_C = \mathfrak{S}_n \cap {}^x\mathfrak{S}_{C^+}$.

*Proof.* (a) is stated, in a general case, in [7, Proposition 3.1]. (b) follows easily from (a). Let us now prove (c). Let $x \in X_C$ and $w \in W_C$. Let $I = \{i \in I_n \mid w(i) < 0\}$ and $J = \{i \in I_n \mid xwx^{-1}(i) < 0\}$, then $\ell_t(w) = |I|$ and $\ell_t(xwx^{-1}) = |J|$, by 2.2. Now let $i \in I$, then $t_i \in W_C$, so $\ell_t(xt_i) > \ell_t(x)$. In other words, $x(i) > 0$. Now, we have $xwx^{-1}(x(i)) = xw(i)$. But, $w(i) < 0$ and $t_{-w(i)} = wt_iw^{-1} \in W_C$. Therefore, $x(-w(i)) = -xw(i) > 0$. This shows that $x(i) \in J$. So, the map $I \to J$, $i \mapsto x(i)$ is well-defined and clearly injective, implying $|I| \le |J|$ as desired.

The last assertion of this proposition follows easily from this inequality and from the fact that $\mathfrak{S}_{C^+} = \{w \in W_C \mid \ell_t(w) = 0\}$. □

**Proposition 2.9.** *Let $C \mathrel{||}\!\!\models n$ and $x \in X_C$ be such that ${}^xS'_C \subset S'_n$. Let $D$ be the unique signed composition of $n$ such that ${}^xS'_C = S'_D$ (see Corollary 2.6). Then $X_C = X_D x$.*

*Proof.* By symmetry, it is sufficient to prove that, if $w \in X_D$, then $wx \in X_C$. Let $\alpha \in \Phi_C^+$. Then, since $x \in X_C$, we have $x(\alpha) \in \Phi_n^+ \cap {}^x\Phi_C = \Phi_D^+$. So $w(x(\alpha)) \in \Phi_n^+$ since $w \in X_D$. So $wx \in X_C$. □

2.6. **Maximal element in $X_C$.** It turns out that, for every signed composition $C$ of $n$, $X_C$ contains a unique element of maximal length (see Proposition 2.12). First, note the following two examples:

(1) if $C$ is parabolic, it is well-known that $\ell_C$ is the restriction of $\ell$ and that, for all $(x, w) \in X_C \times W_C$, we have

$$\ell(xw) = \ell(x) + \ell(w)$$

In particular, $w_n w_C$ is the longest element of $X_C$ (see [9]);

(2) let $C$ be a composition of $n$, then $W_C$ is not in general a standard parabolic subgroup of $W_n$. However, since $W_C$ contains $\mathfrak{T}_n$, $X_C$ is contained in $\mathfrak{S}_n$. This shows that

$$X_C = X_{\overline{C}}^{\bar{n}} = X_{\overline{C}} \cap \mathfrak{S}_n.$$

In particular, $X_C$ contains a unique element of maximal length: this is $\sigma_n \sigma_C$;

Now, let $k$ and $l$ be two non-zero natural numbers such that $k + l = n$. Then $W_{k,l}$ is not a parabolic subgroup of $W_n$. However, $W_{k,\bar{l}}$ is a standard parabolic subgroup of $W_n$ and $W_{k,\bar{l}} \subset W_{k,l}$. So $X_{k,l} \subset X_{k,\bar{l}}$. So, if $x \in X_{k,l}$ and $w \in W_{k,\bar{l}}$, then

$$\ell(xw) = \ell(x) + \ell(w). \tag{2.10}$$

This applies for instance if $w \in W_k \subset W_{k,\bar{l}}$.

Then, we need to introduce a decomposition of $X_C$ using Proposition 2.8 (b). Write $C = (c_1, \dots, c_r) \mathrel{||}\!\!\models n$. We set

$$X_{C,i} = X_{(|c_1|+\cdots+|c_{i-1}|, c_i, \dots, c_r)}^{(|c_1|+\cdots+|c_i|, c_{i+1}, \dots, c_r)}.$$

Then the map

$$\begin{array}{ccc} X_{C,r} \times \cdots \times X_{C,2} \times X_{C,1} & \longrightarrow & X_C \\ (x_r, \dots, x_2, x_1) & \longmapsto & x_r \dots x_2 x_1 \end{array}$$

is bijective by Proposition 2.8 (b). Moreover, by 2.10, we have

$$\ell(x_r \dots x_2 x_1) = \ell(x_r) + \cdots + \ell(x_2) + \ell(x_1) \tag{2.11}$$

for every $(x_r,\dots,x_2,x_1) \in X_{C,r}\times\cdots\times X_{C,2}\times X_{C,1}$. For every $i\in[1,r]$, $X_{C,i}$ contains a unique element of maximal length (see (1)-(2) above). Let us denote it by $\eta_{C,i}$. We set:

$$\eta_C = \eta_{C,r}\dots\eta_{C,2}\eta_{C,1}.$$

Then, by 2.11, we have

**Proposition 2.12.** *Let $C \Vdash n$, then $\eta_C$ is the unique element of $X_C$ of maximal length.*

2.7. **Double cosets representatives.** If $C$ and $D$ are two signed compositions of $n$, we set

$$X_{CD} = X_C^{-1}\cap X_D.$$

**Proposition 2.13.** *Let $C$ and $D$ be two signed composition of $n$ and let $d\in X_{CD}$. Then:*

(a) *There exists a unique signed composition $E$ of $n$ such that $S'_E = S'_C \cap {}^dS'_D$. It will be denoted by $C\cap {}^dD$ or ${}^dD\cap C$. We have $(C\cap {}^dD)^- = C^-\cap {}^dD^-$.*
(b) *$W_C\cap {}^dW_D = W_{C\cap {}^dD}$ and $W_C\cap {}^dS'_D = S'_C\cap {}^dW_D = S'_{C\cap {}^dD}$.*
(c) *If $w\in W_{C\cap {}^dD}$, then $\ell_t(w)=\ell_t(d^{-1}wd)$.*
(d) *If $w\in W_C dW_D$, then there exists a unique pair $(x,y)\in X^C_{C\cap {}^dD}\times W_D$ such that $w=xdy$.*
(e) *Let $(x,y)\in X^C_{C\cap {}^dD}\times W_D$, then $\ell(xdy)\ge \ell(x_S)+\ell_t(x)+\ell(d)+\ell(y_S)+\ell_t(y)$.*
(f) *$d$ is the unique element of $W_CdW_D$ of minimal length.*

*Proof.* (a) follows immediately from Lemma 2.5 (equivalence between (3) and (4)).

(b) It is clear that $W_E\subset W_C\cap {}^dW_D$. Let us show the reverse inclusion. Let $w\in W_C\cap {}^dW_D$. We will show by induction on $\ell_t(w)$ that $w\in W_E$. If $\ell_t(w)=0$, then we see from Proposition 2.8 (d) that $w\in \mathfrak{S}_{C^+}\cap {}^d\mathfrak{S}_{D^+} = \mathfrak{S}_{E^+}$ by definition of $E^+$.

Assume now that $\ell_t(w)>0$ and that, if $w'\in W_C\cap {}^dW_D$ is such that $\ell_t(w')<\ell_t(w)$, then $w'\in W_E$. Since $\ell_t(w)>0$, there exists $i\in[1,n]$ such that $w(i)<0$. In particular, $t_i\in \mathfrak{T}_C$. By the same argument as in the proof of Proposition 2.8 (d), we have that $t_i\in {}^dW_D$. So, $t_i\in T_C\cap {}^dT_D = T_E$. Now, let $w'=wt_i$. Then $t_i\in W_E$, $w'\in W_C\cap {}^dW_D$ and $\ell_t(w')=\ell_t(w)-1$. So, by the induction hypothesis, $w'\in W_E$, so $w\in W_E$.

The other assertions of (b) follow easily.

(c) Let $w=\sigma_1\dots\sigma_l$ be a reduced decomposition of $w$ with respect to $S_C$. Then $d^{-1}wd = (d^{-1}\sigma_1 d)\dots(d^{-1}\sigma_l d)$. But $d^{-1}\sigma_i d\in {}^{d^{-1}}(S'_C\cap {}^dS'_D) = S'_{{}^{d^{-1}}C\cap D}$, so $\ell_t(d^{-1}\sigma_i d)=\ell_t(\sigma_i)$. Since $\ell_t(w)=\ell_t(\sigma_1)+\cdots+\ell_t(\sigma_l)$, we see that $\ell_t(w)\ge \ell_t(d^{-1}wd)$. By symmetry, we obtain the reverse inequality.

(d) Let $w\in W_CdW_D$. Let us write $w=adb$, with $a\in W_C$ and $b\in W_D$. We then write $a=xa'$ with $x\in X^C_{C\cap {}^dD}$ and $a'\in \mathfrak{S}_{C\cap {}^dD}$. Then $d^{-1}a'd\in W_{{}^{d^{-1}}C\cap D}\subset W_D$. Write $y=(d^{-1}a'd)b$. Then $(x,y)\in X^C_{C\cap {}^dD}\times W_D$ and $w=xdy$.

Now let $(x',y')\in X^C_{C\cap {}^dD}\times W_D$ such that $w=x'dy'$. Then $x'^{-1}x = d(yy'^{-1})d^{-1}$. So $x'^{-1}x\in W_{C\cap {}^dD}$, that is $xW_C = x'W_C$. So $x=x'$ and $y=y'$.

(e) Let $(x,y)\in X^C_{C\cap {}^dD}\times W_D$. We will show by induction on $\ell_t(x)+\ell_t(y)$ that

$$\ell(xdy)\ge \ell(x_S)+\ell_t(x)+\ell(d)+\ell(y_S)+\ell_t(y).$$

If $\ell_t(x) = \ell_t(y) = 0$, then $x \in X^{C^-}_{C^- \cap {}^d(D^-)}$, $y \in \mathfrak{S}_{D^+}$ and $d \in X_{C^-,D^-}$. So, by [2, Lemma 2], we have $\ell(xdy) = \ell(x_S) + \ell(d) + \ell(y_S)$, as desired.

Now, let us assume that $\ell_t(x) + \ell_t(y) > 0$ and that the result holds for every pair $(x', y') \in X^C_{C \cap {}^dD} \times W_D$ such that $\ell_t(x') + \ell_t(y') < \ell_t(x) + \ell_t(y)$. By symmetry, and using (c), we can assume that $\ell_t(y) > 0$. So there exists $i \in I_n$ such that $y(i) < 0$. Let $y' = yt_i$. Then $t_i \in T_D$, $\ell(y_S) = \ell(y'_S)$, $\ell_t(y') = \ell_t(y) - 1$. Therefore, by induction hypothesis, we have

$$\ell(xdy') \geq \ell(x_S) + \ell_t(x) + \ell(d) + \ell(y_S) + \ell_t(y) - 1.$$

It is now enough to show that $\ell(xdy't_i) > \ell(xdy')$, that is $xdy'(i) > 0$. Note that $y'(i) > 0$ and that $t_{y'(i)} = y't_iy'^{-1} \in W_D$. So the result follows from the following lemma:

**Lemma 2.14.** *If $d \in X_{CD}$, if $x \in X^D_{C \cap {}^dD}$ and if $j \in [1, n]$ is such that $t_j \in T_D$, then $xd(j) > 0$.*

*Proof.* Since $t_j \in W_D$ and $d \in X_D$, we have $d(j) > 0$. Two cases may occur. If $t_{d(j)} \in T_C$, then $t_{d(j)} = dt_jd^{-1} \in T_{C \cap {}^dD}$. Therefore, $x(d(j)) > 0$ since $x \in X^C_{C \cap {}^dD}$. If $t_{d(j)} \notin T_C$, then $x(d(j)) > 0$ since $x \in W_C = \mathfrak{S}_{C^+} \ltimes \mathfrak{T}_C$. □

(f) follows immediately from (e). □

*Remark 2.15* - Let $C$ and $D$ be two signed compositions of $n$ and let $d \in X_{CD}$. Then $d^{-1} \in X_{DC}$ and, by Proposition 2.9, we have that

$$X_{C \cap {}^dD}\, d = X_{{}^{d^{-1}}C \cap D}.$$

**Corollary 2.16.** *The map $X_{CD} \to W_C \backslash W_n / W_D$ is bijective.*

*Proof.* The proposition 2.13 (f) shows that the map is injective. The surjectivity follows from the fact that, if $w \in W_n$ is an element of minimal length in $W_C w W_D$, then $w \in X_{CD}$. □

**Corollary 2.17.** *If $C$ is parabolic or if $D$ is semi-positive, then*

$$X_D = \coprod_{d \in X_{CD}} X^C_{C \cap {}^dD}\, d.$$

*Proof.* It follows from Corollary 2.16 that

$$|X_D| = |W_n/W_D| = \sum_{d \in X_{CD}} |W_C d W_D / W_D| = \sum_{d \in X_{CD}} |X_{C \cap {}^dD}|,$$

the last equality following from Proposition 2.13 (d). So, it remains to show that, if $d \in X_{CD}$ and if $x \in X^C_{C \cap {}^dD}$, then $xd \in X_D$.

Assume that we have found $s \in S_D$ such that $\ell(xds) < \ell(xd)$. If $s \in T_D$, then $s = t_i$ for som $i \in I_n$. But, by Lemma 2.14, $xd(i) > 0$, so $\ell(xdt_i) > \ell(xd)$, contradicting our hypothesis. Therefore, $s \in S_{D^-}$, that is $s = s_i$ for some $i \in [1, n-1]$. If $C$ is parabolic, $C \cap {}^dD$ is also parabolic. Therefore, $\ell(xds) > \ell(xd)$ which is a contradiction, so $D$ is semi-positive. Therefore, we have that $t_i$ and $t_{i+1}$ belong to

$T_D$. Thus, by Lemma 2.14, we have $xd(i) > 0$ and $xd(i+1) > 0$. Moreover, since $\ell(xds_i) < \ell(xd)$, we have

$$(*) \qquad 0 < xd(i+1) < xd(i).$$

But, since $d \in X_D$, we have $d(i+1) > d(i)$. So, by Proposition 2.13 (b), we have that $ds_id^{-1} \in S_{C\cap {}^dD}$. Thus $\ell(x(ds_id^{-1})) > \ell(x)$ because $x \in X^C_{C\cap {}^dD}$. In other words, $xd(i+1) > xd(i)$. This contradicts $(*)$. $\square$

If $E$ is a signed composition of $n$ such that $C \subset E$ and $D \subset E$, we set $X^E_{CD} = X_{CD} \cap W_E$.

*Example.* It is not true in general that $X_D = \coprod_{d\in X_{CD}} X^C_{C\cap {}^dD} d$. This is false, if $n = k + l$ with $k$, $l \geq 1$, $C = (\bar{k}, \bar{l})$ and $D = (n)$. See Example 2.25 for precisions.

In [2], the authors has given a proof of the Solomon theorem using tools which sound like the above results. Here, we cannot translate their proof because of the complexity of the decomposition of $X_D$ (which involve negative coefficients).

2.8. **A partition of $W_n$.** If $C = (c_1, \ldots, c_r)$ is a signed composition of $n$, we set

$$A_C = \{s_{|c_1|+\cdots+|c_i|} \mid i \in [1, r] \text{ and } c_i < 0 \text{ and } c_{i+1} > 0\}$$

and

$$\mathcal{A}_C = S'_C \coprod A_C.$$

As example, $A_{(1,\bar{3},\bar{1},2,\bar{1},1)} = \{s_5, s_8\}$. Note that $\mathcal{A}_C = \mathcal{A}_D$ if and only if $C = D$. If $w \in W_n$, then we define *the ascent set of* $w$:

$$\mathcal{U}'_n(w) = \{s \in S'_n \mid \ell(ws) > \ell(w)\}.$$

Finally, following Mantaci-Reutenauer, we associate to each element $w \in W_n$ a signed composition $\boldsymbol{C}(w)$ as follows. First, let $\boldsymbol{C}^+(w)$ denote the biggest composition (for the order $\subset$) of $n$ such that, for every $1 \leq i \leq r$, the map $w : I^{(i)}_{\boldsymbol{C}^+(w)} \to I_n$ is increasing and has constant sign. Now, we define $\nu_i = \operatorname{sign}(w(j))$ for $j \in I^{(i)}_{\boldsymbol{C}^+(w)}$. The *descent composition* of $w$ is $\boldsymbol{C}(w) = (\nu_1 c^+_1, \ldots, \nu_r c^+_r)$.

*Example.* $\boldsymbol{C}(\underbrace{9.\bar{3}\bar{2}\bar{1}.\bar{4}.58.\bar{6}.7}_{\in W_9}) = (1, \bar{3}, \bar{1}, 2, \bar{1}, 1) \mathrel{||}\!\!= 9.$

The following proposition is easy to check (see Remark 2.1):

**Proposition 2.18.** *If $w \in W_n$, then $\mathcal{U}'_n(w) = \mathcal{A}_{\boldsymbol{C}(w)}$.*

*Remark.* Mantaci and Reutenauer have defined the *descent shape* of a signed permutation [16]. It is a signed composition defined similarly than descent composition except that the absolute value of the letters in $u_i$ must be in increasing order. For instance, the descent shape of $9.\bar{3}.\bar{2}.\bar{1}\bar{4}.58.\bar{6}.7$ is $(1, \bar{1}, \bar{1}, \bar{2}, 2, \bar{1}, 1)$.

*Example 2.19* - Let $n'$ be a non-zero natural number, $n' < n$ and let $c \in \mathbb{Z}$ such that $n - n' = |c|$. Let $w \in W_{n'} \subset W_n$ and write $\boldsymbol{C}(w) = (c_1, \ldots, c_r) \mathrel{||}\!\!= n'$. Then $\boldsymbol{C}(\eta_{(n',c)}w) = (c_1, \ldots, c_r, c)$. Consequently, if $C \mathrel{||}\!\!= n$, an easy induction argument shows that $\boldsymbol{C}(\eta_C) = C$.

We have then defined a surjective map

$$\boldsymbol{C} : W_n \longrightarrow \operatorname{Comp}(n)$$

whose fibers are equal to those of the application $\mathcal{U}'_n : W_n \to \mathcal{P}(S'_n)$. The surjectivity follows from Example 2.19. If $C \,\|\!\!\models n$, we define

$$Y_C = \{w \in W_n \mid \boldsymbol{C}(w) = C\}.$$

Then

$$W_n = \coprod_{C \,\|\!\models n} Y_C.$$

*Example 2.20* - We have $Y_n = \{1_n\}$, $Y_{\bar{n}} = \{\sigma_n w_n\}$, $Y_{(1,\dots,1)} = \{\sigma_n\}$ and $Y_{(\bar{1},\dots,\bar{1})} = \{w_n\}$.

First, note the following elementary facts.

**Lemma 2.21.** *Let $C$ and $D$ be two signed compositions of $n$. Then:*

(a) *If $Y_C \cap X_D \neq \varnothing$, then $Y_C \subset X_D$.*
(b) *$\eta_C \in Y_C$ and $Y_C \subset X_C$.*

*Proof.* (a) If $w \in W_n$, then $w \in X_D$ if and only if $\mathcal{U}'_n(w)$ contains $S'_D$. Since the map $w \mapsto \mathcal{U}'_n(w)$ is constant on $Y_C$ (see Proposition 2.18), (a) follows.

(b) By Example 2.19, we have $\eta_C \in Y_C \cap X_C$. Therefore, by (a), $Y_C \subset X_C$. $\square$

We then define a relation $\leftarrow$ between signed composition of $n$ as follow. If $C$, $D \,\|\!\!\models n$, we write $C \leftarrow D$ if $Y_D \subset X_C$. We denote by $\preccurlyeq$ the transitive closure of the relation $\leftarrow$. It follows from Lemma 2.21 (a) that

$$X_C = \coprod_{C \leftarrow D} Y_D. \tag{2.22}$$

*Example 2.23* - Let $w \in W_n$. By Remark 2.1, $w \in X_{\bar{n}}$ if and only if the sequence $(w(1), w(2), \dots, w(n))$ of elements of $I_n$ is strictly increasing (see Remark 2.1). So there exists a unique $k \in \{0, 1, 2, \dots, n\}$ such that $w(i) > 0$ if and only if $i > k$. Note that $k = \ell_t(w)$. Let $i_1 < \dots < i_k$ be the sequence of elements of $I_n$ such that $(w(1), \dots, w(k)) = (\bar{i}_k, \dots, \bar{i}_1)$. Then $w = r_{i_1} r_{i_2} \dots r_{i_k}$ where, if $1 \le i \le n$, we set $r_i = s_{i-1} \dots s_2 s_1 t$. Note that $\boldsymbol{C}(w) = (\bar{k}, n-k)$. Therefore,

$$X_{\bar{n}} = \{r_{i_1} r_{i_2} \dots r_{i_k} \mid 0 \le k \le n \text{ and } 1 \le i_1 < i_2 < \dots < i_k \le n\}.$$

Note that $\ell(r_{i_1} r_{i_2} \dots r_{i_k}) = i_1 + i_2 + \dots + i_k$ and $\ell_t(r_{i_1} r_{i_2} \dots r_{i_k}) = k$. We get

$$X_{\bar{n}} = \coprod_{0 \le k \le n} Y_{(\bar{k}, n-k)},$$

and, for every $k \in \{0, 1, 2, \dots, n\}$, we have

$$Y_{(\bar{k}, n-k)} = \{r_{i_1} r_{i_2} \dots r_{i_k} \mid 1 \le i_1 < i_2 < \dots < i_k \le n\}.$$

This shows that $(\bar{n}) \leftarrow (\bar{k}, n-k)$.

**Proposition 2.24.** *Let $C$ and $D$ be two signed compositions of $n$. Then:*

(a) *$C \leftarrow C$.*
(b) *If $C \subset D$, then $C \leftarrow D$.*
(c) *$\preccurlyeq$ is an order on* $\mathrm{Comp}(n)$.

*Proof.* (a) follows immediately from Lemma 2.21 (b).

(b) If $C \subset D$, then $X_D \subset X_C$. But, by Lemma 2.21 (b), we have $Y_D \subset X_D$. So $C \leftarrow D$.

(c) Let $a_C = \ell(\mu_C)$. By (a), $\preccurlyeq$ is reflexive. By definition, it is transitive. So it is sufficient to show that it is antisymmetric. But it follows from Lemma 2.21 (b) that:

- If $C \leftarrow D$, then $a_D \leq a_C$.
- If $C \leftarrow D$ and if $a_C = a_D$, then $C = D$.

The assertion (c) now follows easily from these two remarks. $\square$

*Example 2.25* - If $C = (c_1, \ldots, c_r)$ is a composition of $n$ (not a signed composition), we will prove that

$$X_{\bar{n}} = \coprod_{\substack{0 \leq m_2 \leq c_2 \\ 0 \leq m_3 \leq c_3 \\ \cdots \\ 0 \leq m_r \leq c_r}} X^C_{(\bar{c}_1, m_2, c_2 - m_2, \ldots, m_r, c_r - m_r)} Y^{(\bar{c}_1, m_2, c_2 - m_2, \ldots, m_r, c_r - m_r)}_{(\bar{c}_1, \bar{m}_2, c_2 - m_2, \ldots, \bar{m}_r, c_r - m_r)} \sigma^{-1}_{C, m_2, \ldots, m_r},$$

where $\sigma_{C,m_2,\ldots,m_r} \in \mathfrak{S}_n$ satisfies

$$\sigma_{C,m_2,\ldots,m_r}(S'_{(c_1, m_2, c_2 - m_2, \ldots, m_r, c_r - m_r)}) \subset S'_n$$

and
$$\sigma_{C,m_2,\ldots,m_r} \in X_{(c_1, m_2, c_2 - m_2, \ldots, m_r, c_r - m_r)}.$$

By an easy induction argument, it is sufficient to prove it whenever $r = 2$. In other words, we want to prove that, if $k + l = n$ with $k,\, l \geq 0$, then

$$(*) \qquad X_{\bar{n}} = \coprod_{0 \leq m \leq l} X^{(\bar{k}, \bar{l})}_{(\bar{k}, m, l-m)} Y^{(\bar{k}, m, l-m)}_{(\bar{k}, \bar{m}, l-m)} \sigma^{-1}_{k,l,m},$$

where $\sigma_{k,l,m} \in \mathfrak{S}_n$ satisfies $\sigma_{k,l,m}(S'_{k,m,l-m}) \subset S'_n$ and $\sigma_{k,l,m} \in X_{(k,m,l-m)}$. But, if $0 \leq m \leq l$, we set

$$\sigma_{k,l,m}(i) = \begin{cases} m + i & \text{if } 1 \leq i \leq k, \\ i - k & \text{if } k + 1 \leq i \leq k + m, \\ i & \text{if } k + m + 1 \leq i \leq n, \end{cases}$$

and one can easily check that $(*)$ holds. Moreover, since $S'_{k,m,l-m} = S'_n \backslash \{s_k, s_{k+m}\}$, we get that $\sigma_{k,l,m}(S'_{k,m,l-m}) \subset S'_n$ and $\sigma_{k,l,m} \in X_{(k,m,l-m)}$.

## 3. Generalized descent algebra

3.1. **Definition.** If $C$ and $D$ are two signed compositions of $n$ such that $C \subset D$, we set

$$x^D_C = \sum_{w \in X^D_C} w \qquad \in \mathbb{Z}W_D$$

and
$$y^D_C = \sum_{w \in Y^D_C} w \qquad \in \mathbb{Z}W_D.$$

Now, let

$$\Sigma'(W_D) = \mathop{\oplus}_{C \subset D} \mathbb{Z} y^D_C \qquad \subset \mathbb{Z}W_D.$$

Note that

$$\Sigma'(W_D) = \mathop{\oplus}_{C \subset D} \mathbb{Z} x_C^D$$

by 2.22 and Proposition 2.24. We define

$$\theta_D : \Sigma'(W_D) \longrightarrow \mathbb{Z} \operatorname{Irr} W_D$$

as the unique $\mathbb{Z}$-linear map such that

$$\theta_D(x_C^D) = \operatorname{Ind}_{W_C}^{W_D} 1_C$$

for every $C \subset D$. Here, $1_C$ is the trivial character of $W_C$. We denote by $\varepsilon_D$ the sign character of $W_D$.

*Notation.* If $D = (n)$, we set $x_C^D = x_C$, $y_C^D = y_C$ for simplification. If $E$ is $\mathbb{Z}$-module, we denote by $\mathbb{Q}E$ the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} E$. We denote by $\theta_{D,\mathbb{Q}}$ the extension of $\theta_D$ to $\mathbb{Q}\Sigma'(W_D)$ by $\mathbb{Q}$-linearity.

*Remark.* $\Sigma'(W_n)$ contains the Solomon descent algebras of $W_n$ and $\mathfrak{S}_n$. Moreover, $\Sigma'(W_n)$ is precisely the *Mantaci-Reutenauer algebra* which is, by definition, generated by $y_D = y_D^{(n)}$, for all $D \mathrel{|\!\!\models} n$.

3.2. **First properties of $\boldsymbol{\theta_D}$.** By the Mackey formula for product of induced characters and by Proposition 2.13, we have that

$$\textbf{(3.1)} \qquad \theta_n(x_C)\theta_n(x_D) = \sum_{d \in X_{CD}} \theta_n(x_{{}^{d^{-1}}C \cap D}).$$

*Example 3.2* - If $C$ is parabolic or $D$ is semi-positive, then, by Corollary 2.17, we have

$$x_D = \sum_{d \in X_{CD}} x^C_{C \cap {}^d D} d.$$

Therefore, by Proposition 2.8 (b) and Remark 2.15, we get

$$x_C x_D = \sum_{d \in X_{CD}} x_{{}^{d^{-1}}C \cap D}.$$

So $x_C x_D \in \Sigma'(W_n)$ and, by 3.1, $\theta_n(x_C x_D) = \theta_n(x_C)\theta_n(x_D)$. $\square$

Before starting the proof of the fact that $\Sigma'(W_D)$ is a subalgebra of $\mathbb{Z}W_D$ and that $\theta_D$ is a morphism of algebras, we need the following result, which will be useful for arguing by induction. If $C \subset D$, the transitivity of induction and Proposition 2.8 (b) show that the diagram

$$\textbf{(3.3)} \qquad \begin{tikzcd} \Sigma'(W_C) \arrow[r, "x_C^D \cdot"] \arrow[d, "\theta_C"'] & \Sigma'(W_D) \arrow[d, "\theta_D"] \\ \mathbb{Z} \operatorname{Irr} W_C \arrow[r, "\operatorname{Ind}_{W_C}^{W_D}"] & \mathbb{Z} \operatorname{Irr} W_D \end{tikzcd}$$

is commutative.

Now, let $p_D : W_D \to \mathfrak{S}_{D^+}$ be the canonical projection. It induces an injective morphism of $\mathbb{Z}$-algebras $p_D^* : \mathbb{Z} \operatorname{Irr} \mathfrak{S}_{D^+} \to \mathbb{Z} \operatorname{Irr} W_D$. Moreover, the algebra

$\Sigma'(\mathfrak{S}_{D^+})$ coincides with the usual descent algebra in symmetric groups and is contained in $\Sigma'(W_D)$. Also, the diagram

**(3.4)**
$$\begin{tikzcd} \Sigma'(\mathfrak{S}_{D^+}) \arrow[r, hook] \arrow[d, "\theta_{D^-}"'] & \Sigma'(W_D) \arrow[d, "\theta_D"] \\ \mathbb{Z}\operatorname{Irr}\mathfrak{S}_{D^+} \arrow[r, hook, "p_D^*"] & \mathbb{Z}\operatorname{Irr}W_D \end{tikzcd}$$

is commutative.

*Example 3.5 -* We have $y_{(\bar{1},\ldots,\bar{1})} = w_n$, $y_{\bar{n}} = w_n\sigma_n = \sigma_n w_n$, $y_n = 1$ and $y_{(1,\ldots,1)} = \sigma_n$. It is well-known [18] that $y_{(\bar{1},\ldots,\bar{1})}$ belongs to the classical descent algebra of $W_n$ and that

(a) $$\theta_n(w_n) = \varepsilon_n.$$

On the other hand,

(b) $$\theta_n(1_n) = 1_{(n)}.$$

Also, by the commutativity of the diagram 3.4 and as above, we have

(c) $$\theta_n(\sigma_n) = \gamma_n,$$

where $\gamma_n = p_n^*\varepsilon_{\bar{n}}$. Finally, $w_n$ is a $\mathbb{Z}$-linear combination of $x_C$, where $C$ runs over the parabolic compositions of $n$. Therefore, by Example 3.2, we have, for every $x \in \Sigma'(W_n)$,

(d) $$\theta_n(w_n x) = \theta_n(w_n)\theta_n(x) = \varepsilon_n\theta_n(x).$$

In particular,

(e) $$\theta_n(y_{\bar{n}}) = \theta_n(w_n\sigma_n) = \varepsilon_n\gamma_n.$$

So we have obtained the four linear characters of $W_n$ as images by $\theta_n$ of explicit elements of $\Sigma'(W_n)$.

Let $\deg_D : \mathbb{Z}\operatorname{Irr}W_D \to \mathbb{Z}$ be the $\mathbb{Z}$-linear map sending an irreducible character of $W_D$ to its degree. It is a morphism of $\mathbb{Z}$-algebras. Let $\operatorname{aug}_D : \mathbb{Z}W_D \to \mathbb{Z}$ be the augmentation morphism, then it is clear that the diagram

**(3.6)**
$$\begin{tikzcd} \Sigma'(W_D) \arrow[r, "\theta_D"] \arrow[dr, "\operatorname{aug}_D"'] & \mathbb{Z}\operatorname{Irr}W_D \arrow[d, "\deg_D"] \\ & \mathbb{Z} \end{tikzcd}$$

is commutative.

3.3. **Main result.** We are now ready to prove that $\Sigma'(W_D)$ is a $\mathbb{Z}$-subalgebra of $\mathbb{Z}W_D$ and that $\theta_D$ is a surjective morphism of algebras.

**Theorem 3.7.** *Let $D$ be a signed composition of $n$, then:*

(a) *$\Sigma'(W_D)$ is a $\mathbb{Z}$-subalgebra of $\mathbb{Z}W_D$;*

(b) *$\theta_D$ is a morphism of algebra;*

(c) *$\theta_D$ is surjective and $\operatorname{Ker}\theta_D = \bigoplus_{\substack{C,C'\subset D \\ C\equiv_D C'}} \mathbb{Z}(x_C^D - x_{C'}^D)$;*

(d) *$\operatorname{Ker}\theta_{D,\mathbb{Q}}$ is the radical of the algebra $\mathbb{Q}\Sigma'(W_D)$. Moreover, $\mathbb{Q}\Sigma'(W_D)$ is a split algebra whose largest semisimple quotient is commutative. In particular, all its simple modules are of dimension 1.*

*Proof.* We want to prove the theorem by induction on $|W_D|$. By taking direct products, we may therefore assume that $D = (n)$ or $D = (\bar{n})$. If $D = (\bar{n})$, then it is well-known that (a), (b), (c) and (d) hold. So we may assume that $D = (n)$ and that (a), (b), (c) and (d) hold for every signed composition $D'$ of $n$ different from $(n)$.

(a) and (b): Let $A$ and $B$ be two signed compositions of $n$. We want to prove that $x_A x_B \in \Sigma'(W_n)$ and that $\theta_n(x_A x_B) = \theta_n(x_A)\theta_n(x_B)$. If $A$ is parabolic or $B$ is semi-positive, then this is just Example 3.2. So we may assume that $A$ is not parabolic and $B$ is not semi-positive.

First, note that $B \subset B^+$ and that $B^+$ is semi-positive. Therefore, by Proposition 2.8(b) and Example 3.2, we have

$$x_A x_B = x_A x_{B^+} x_B^{B^+} = \sum_{d\in X_{A,B^+}} x_{{}^{d^{-1}}A\cap B^+} x_B^{B^+} = x_{B^+} \sum_{d\in X_{A,B^+}} x^{B^+}_{{}^{d^{-1}}A\cap B^+} x_B^{B^+}.$$

Assume first that $B^+ \neq (n)$. Then, by induction hypothesis,

$$\sum_{d\in X_{A,B^+}} x^{B^+}_{{}^{d^{-1}}A\cap B^+} x_B^{B^+} \in \Sigma'(\mathfrak{S}_{B^+})$$

and

$$\theta_{B^+}\Big(\sum_{d\in X_{A,B^+}} x^{B^+}_{{}^{d^{-1}}A\cap B^+} x_B^{B^+}\Big) = \sum_{d\in X_{A,B^+}} \theta_{B^+}(x^{B^+}_{{}^{d^{-1}}A\cap B^+})\theta_{B^+}(x_B^{B^+}).$$

Therefore, by 3.3 and 3.4, $x_A x_B \in x_{B^+}\Sigma'(\mathfrak{S}_{B^+}) \subset \Sigma'(\mathfrak{S}_n) \subset \Sigma'(W_n)$ and, by 3.3 and by the Mackey formula for tensor product, we get

$$\begin{aligned}
\theta_n(x_A x_B) &= \operatorname{Ind}_{W_{B^+}}^{W_n}\Big(\sum_{d\in X_{A,B^+}} \theta_{B^+}(x^{B^+}_{{}^{d^{-1}}A\cap B^+})\theta_{B^+}(x_B^{B^+})\Big) \\
&= \theta_n(x_A)\operatorname{Ind}_{W_{B^+}}^{W_n} \theta_{B^+}(x_B^{B^+}) \\
&= \theta_n(x_A)\theta_n(x_B),
\end{aligned}$$

as desired.

Therefore, it remains to consider the case where $B^+ = (n)$. In particular, $B = (n)$ or $(\bar{n})$. Since $B$ is not semi-positive, we have $B = (\bar{n})$. By Example 2.25, we have

$$x_{\bar{n}} = x_{A^-}^{A^+} + \sum_{D\subset A^+} a_D x_D^{A^+}(\sigma_D^{-1} - 1)$$

where $a_D \in \mathbb{Z}$ and $\sigma_D(S'_D) \subset S'_n$ and $\sigma_D \in X_D$ for every $D \subset A^+$. Therefore,

$$x_A x_B = x_A x_{\bar{n}} = x_{A^+}\Big(x_A^{A^+} x_{A^-}^{A^+} + \sum_{D \subset A^+} a_D x_A^{A^+} x_D^{A^+}(\sigma_D^{-1} - 1)\Big).$$

Now, $W_A$ is a standard parabolic subgroup of $W_{A^+}$. So, by Example 3.2, we have

$$x_A x_B = \sum_{d \in X_{A,A^-}^{A^+}} x_{{}^{d^{-1}}A \cap (A^-)} + \sum_{D \subset A^+} \Big(a_D \sum_{d \in X_{A,D}^{A^+}} x_{{}^{d^{-1}}A \cap D}(\sigma_D^{-1} - 1)\Big).$$

Therefore, since $\sigma_D(S'_D) \subset S'_n$ and $\sigma_D \in X_D$, we have that $x_{{}^{d^{-1}}A\cap D}\sigma_D^{-1} = x_{{}^{\sigma_D}({}^{d^{-1}}A \cap D)}$. So $x_A x_B \in \Sigma'(W_n)$ and $\theta_n(x_A x_B) = \theta_n(x_A)\theta_n(x_B)$ by the Mackey formula for tensor product of induced characters. This concludes the proof of (a) and (b). Indeed, the surjectivity of $\theta_n$ is well-known.

(c) First, let us show that $\theta_n$ is surjective. Using the induction hypothesis, the commutativity of the diagram 3.3, and the classical description of irreducible characters of $W_n$, we are reduced to prove that, for every $\chi \in \operatorname{Irr} \mathfrak{S}_n$, $p_n^*(\chi)$ and $p_n^*(\chi)\varepsilon_n$ lie in the image of $\theta_n$. But it is well-known that $\theta_{\bar{n}}$ is surjective. So the result follows from the commutativity of the diagram 3.4 and from Example 3.5 (d).

Now, let $I = \sum_{\substack{C,C' \,|\!\models n \\ C \equiv_n C'}} \mathbb{Z}(x_C - x_{C'})$. Then it is clear that $I \subset \operatorname{Ker}\theta_n$. Let $J = \oplus_{\lambda \in \operatorname{Bip}(n)} \mathbb{Z} x_{\hat{\lambda}}$. Then $\Sigma'(W_n) = I \oplus J$ and the map $\theta_n : J \to \mathbb{Z}\operatorname{Irr} W_n$ is surjective. Since $J$ and $\mathbb{Z}\operatorname{Irr} W_n$ have the same rank (equal to $|\operatorname{Bip}(n)|$), we get that $J \cap \operatorname{Ker}\theta_n = 0$. So $I = \operatorname{Ker}\theta_n$.

(d) Let $R = \operatorname{Rad}(\mathbb{Q}\Sigma'(W_n))$ and $K = \operatorname{Ker}\theta_{n,\mathbb{Q}}$. Since $\operatorname{Im}(\theta_{n,\mathbb{Q}}) = \mathbb{Q}\operatorname{Irr} W_n$ is a semisimple algebra, we get that $R \subset K$.

Now, let $\chi : \mathbb{Q}W_n \to \mathbb{Q}$ be the character of the $\mathbb{Q}W_n$-module $\mathbb{Q}W_n$ (the regular representation). Then, $\chi(w) = 0$ for every $w \neq 1$. Let $\chi'$ denote the restriction of $\chi$ to $\Sigma'(W_n)$. We have $\chi'(x_C) = \chi(1)$ for every $C \,|\!\models n$. Therefore, $\chi'(x) = 0$ for every $x \in K$ by (c). We fix now $x \in K$. Then, for every $y \in \mathbb{Q}\Sigma'(W_n)$, we have $\chi'(xy) = 0$ because $xy \in K$ by (b). Since the $\mathbb{Q}\Sigma'(W_n)$-module $\mathbb{Q}W_n$ is faithful, this implies that $x \in R$. So $K \subset R$. □

*Remark.* $\Sigma'(W_C) \simeq \Sigma'(W_{c_1}) \otimes \Sigma'(W_{c_2}) \otimes \cdots \otimes \Sigma'(W_{c_r})$.

3.4. **Further properties of $\boldsymbol{\theta_D}$.** Let $\tau_D : \mathbb{Z}W_D \to \mathbb{Z}$ be the unique linear map such that $\tau_D(w) = 0$ if $w \neq 1$ and $\tau_D(1) = 1$. Then $\tau_D$ is the canonical symmetrizing form on $\mathbb{Z}W_D$: in particular, the map $\mathbb{Z}W_D \times \mathbb{Z}W_D \to \mathbb{Z}$, $(x,y) \mapsto \tau_D(xy)$ is a non-degenerate symmetric bilinear form on $\mathbb{Z}W_D$. We denote by $\langle\, .\,,.\, \rangle_D$ the scalar product on $\mathbb{Z}\operatorname{Irr} W_D$ such that $\operatorname{Irr} W_D$ is an orthonormal basis. The following property is a kind of "isometry property" for the morphism $\theta_D$.

**Proposition 3.8.** *If $x$, $y \in \Sigma'(W_D)$, then $\tau_D(xy) = \langle\, \theta_D(x), \theta_D(y)\, \rangle_D$.*

*Proof.* Let $C$ and $C'$ be two signed compositions of $n$ such that $C$, $C' \subset D$. Then $\tau_D(x_C^D x_{C'}^D) = |X_{CC'}^D|$ by definition of $\tau_D$. Moreover, since $\theta_D(x_C)$ and $\theta_D(x_{C'})$ take only rational values, we have

$$\langle\, \theta_D(x_C^D), \theta_D(x_{C'}^D)\, \rangle_D = \langle\, \theta_D(x_C^D)\theta_D(x_{C'}^D), 1_{W_D}\, \rangle_D.$$

But, by 3.1 and by Frobenius reciprocity, we have

$$\langle\, \theta_D(x_C^D)\theta_D(x_{C'}^D), 1_{W_D}\,\rangle_D = |X_{CC'}^D|.$$

So the proposition follows now from the fact that $(x_C^D)_{C\subset D}$ generates $\Sigma'(W_D)$. $\square$

**Corollary 3.9.** $\operatorname{Ker}\theta_D = \{x \in \Sigma'(W_D) \mid \forall\ y \in \Sigma'(W_D),\ \tau_D(xy) = 0\}$.

*Proof.* Since $\langle\, .\,,.\,\rangle_D$ is non-degenerate on $\mathbb{Z}\operatorname{Irr} W_D$, this follows from Proposition 3.8. $\square$

Write $D = (d_1, \dots, d_r)$ and let $\operatorname{Bip}(D)$ denote the set of $r$-uples $(\lambda_{(1)}, \dots, \lambda_{(r)})$ of bipartitions $\lambda_{(i)} = (\lambda_{(i)}^+, \lambda_{(i)}^-)$ such that $\lambda_{(i)}^- = \emptyset$ if $d_i < 0$ and $|\lambda_{(i)}| = |d_i|$ for every $i \in [1, r]$. If $\lambda \in \operatorname{Bip}(D)$, we denote by $\mathcal{C}_\lambda^D$ the conjugacy class in $W_D$ of a Coxeter element of $W_{\hat\lambda}$ (with respect to $S_{\hat\lambda}$). Let $f_\lambda^D$ denote the characteristic function of $\mathcal{C}_\lambda^D$. Then $f_\lambda$ is a primitive idempotent of $\mathbb{Q}\operatorname{Irr} W_n$. Moreover, $(f_\lambda^D)_{\lambda\in\operatorname{Bip}(D)}$ is a complete family of orthogonal primitive idempotents of $\mathbb{Q}\operatorname{Irr} W_D$. Since $\theta_D$ is surjective, there exists a family of idempotents $(E_\lambda^D)_{\lambda\in\operatorname{Bip}(D)}$ of $\mathbb{Q}\Sigma'(W_D)$ such that

(1) $\forall\ \lambda \in \operatorname{Bip}(D)$, $\theta_D(E_\lambda^D) = f_\lambda^D$.
(2) $\forall\ \lambda, \mu \in \operatorname{Bip}(D)$, $\lambda \neq \mu \Rightarrow E_\lambda^D E_\mu^D = E_\mu^D E_\lambda^D = 0$.
(3) $\displaystyle\sum_{\lambda\in\operatorname{Bip}(D)} E_\lambda^D = 1$.

**Proposition 3.10.** *If $x \in \Sigma'(W_D)$, then*

$$\theta_D(x) = |W_D| \sum_{\lambda \in \Vdash D} \frac{\tau_D(xE_\lambda^D)}{|\mathcal{C}_\lambda^D|} f_\lambda^D \in \mathbb{Z}\operatorname{Irr} W_D.$$

*Proof.* If $f \in \mathbb{Q}\operatorname{Irr} W_D$, then

$$f = |W_D| \sum_{\lambda\in\operatorname{Bip}(D)} \frac{\langle\, f, f_\lambda^D\,\rangle_D}{|\mathcal{C}_\lambda^D|} f_\lambda^D \in \mathbb{Z}\operatorname{Irr} W_D.$$

If $f = \theta_D(x)$ with $x \in \Sigma'(W_D)$, then we get the desired formula just by applying Proposition 3.8 and the property (1) above. $\square$

3.5. **Character table.** Since all irreducible characters of $W_D$ have rational values, the algebra $\mathbb{Q}\operatorname{Irr} W_D$ may be identified with the $\mathbb{Q}$-algebra of central functions $W_D \to \mathbb{Q}$. If $\lambda \in \operatorname{Bip}(D)$, we denote by $\operatorname{ev}_\lambda^D : \mathbb{Q}\operatorname{Irr} W_D \to \mathbb{Q}$, $\chi \mapsto \chi(c_\lambda^D)$, where $c_\lambda^D$ is some element of $\mathcal{C}_\lambda^D$ (for instance, a Coxeter element of $W_{\hat\lambda}$). Then $\operatorname{ev}_\lambda^D$ is a morphism of algebras: it is an irreducible representation of $\mathbb{Q}\operatorname{Irr} W_D$. Moreover, $\{\operatorname{ev}_\lambda^D \mid \lambda \in \operatorname{Bip}(D)\}$ is a complete set of representatives of isomorphy classes of irreducible representations of $\mathbb{Q}\operatorname{Irr} W_D$. Now, let $\mathbb{Q}_\lambda^D$ denote the $\mathbb{Q}\Sigma'(W_D)$-module whose underlying vector space is $\mathbb{Q}$ and on which an element $x \in \mathbb{Q}\Sigma'(W_D)$ acts by multiplication by $\pi_\lambda^D(x) = (\operatorname{ev}_\lambda^D \circ \theta_D)(x)$. Then, by Theorem 3.7, we get:

**Proposition 3.11.** $\{\mathbb{Q}_\lambda^D \mid \lambda \in \operatorname{Bip}(D)\}$ *is a complete set of isomorphy classes of $\mathbb{Q}\Sigma'(W_D)$-modules. We have*

$$\operatorname{Irr}(\mathbb{Q}\Sigma'(W_D)) = \{\pi_\lambda^D \mid \lambda \in \operatorname{Bip}(D)\}.$$

The *character table* of $\mathbb{Q}\Sigma'(W_D)$ is the square matrix whose rows and the columns are indexed by $\mathrm{Bip}(D)$ and whose $(\lambda,\mu)$-entry is the value of the irreducible character $\pi_\lambda^D(x_{\hat{\mu}}^D)$. Note that

$$\pi_\lambda^D(x_{\hat{\mu}}^D) = \left(\mathrm{Ind}_{W_{\hat{\mu}}}^{W_D} 1_{\hat{\mu}}\right)(c_\lambda^D).$$

*Notation.* If $D=(n)$, we denote $\mathcal{C}_\lambda^D$, $f_\lambda^D$, $E_\lambda^D$, $c_\lambda^D$, $\mathrm{ev}_\lambda^D$, $\mathbb{Q}_\lambda^D$ and $\pi_\lambda^D$ by $\mathcal{C}_\lambda$, $f_\lambda$, $E_\lambda$, $c_\lambda$, $\mathrm{ev}_\lambda$, $\mathbb{Q}_\lambda$ and $\pi_\lambda$ respectively.

Now, if $\lambda$, $\mu \in \mathrm{Bip}(D)$, we write $\lambda \subset \mu$ if there exists some $w \in W_D$ such that $W_{\hat{\lambda}} \subset {}^w W_{\hat{\mu}}$. By Proposition 2.7, $\subset$ is a partial order on $\mathrm{Bip}(D)$. For this partial order, the character table of $\mathbb{Q}\Sigma'(W_D)$ is triangular :

**Proposition 3.12.** *If $\pi_\lambda^D(x_{\hat{\mu}}^D) \neq 0$, then $\lambda \subset \mu$.*

*Proof.* We may, and we will, assume that $D=(n)$. If $\pi_\lambda(x_{\hat{\mu}}) \neq 0$, then there exists $w \in W_n$ such that $wc_\lambda w^{-1} \in W_{\hat{\mu}}$. Therefore, there exists $\nu \in \mathrm{Bip}(\hat{\mu})$ and $w' \in W_{\hat{\mu}}$ such that $w'wc_\lambda w^{-1}w'^{-1}$ is a Coxeter element of $W_{\hat{\nu}}$. Let $C$ denote the unique signed composition of $n$ such that $W_{\hat{\nu}} = W_C$ and let $\lambda' = \boldsymbol{\lambda}(C)$. Then $w'wc_\lambda w^{-1}w'^{-1}$ is conjugate to $c_{\lambda'}$. Therefore, $\lambda = \lambda'$. This completes the proof of the proposition. □

In the last section of this paper, we will give the character table of $\Sigma'(W_2)$.

3.6. **Combinatorial description.** In $\mathfrak{S}_n$, the refinement of compositions is useful to construct $X_C$ from $Y_D$ without considering subsets of $S'_{\bar{n}}$. The aim of this part is to describe such a procedure in our case. Start with an example, consider $C=(\bar{2},1)$, then the subsets of $S'_3$ containing $S_C=\{s_1,t_3\}$ are $\{s_1,s_2,t_3\}=\mathcal{A}_{(\bar{2},1)}$; $\{s_1,t_2,t_3\}=\mathcal{A}_{(\bar{1},1,1)}$; $\{s_1,t_1,t_2,t_3\}=\mathcal{A}_{(2,1)}$, $\{s_1,s_2,t_2,t_3\}=\mathcal{A}_{(\bar{1},\bar{2})}$ and $S'_3=\mathcal{A}_{(3)}$. Observe that $(1,2)$ (which corresponds to $\{s_2,t_1,t_2,t_3\} \not\supset S_C$) is not obtained. Here, we define a procedure which give $(\bar{2},1)$, $(\bar{1},1,1)$, $(\bar{1},2)$ and $(3)$, without to obtain $(1,2)$ .

Let $C=(c_1,\ldots,c_k) \,|\!\!\models n$, we write:

- $C \xleftarrow{B} D$ if $D=(a_1,b_1,a_2,b_2,\ldots,a_k,b_k) \,|\!\!\models n$ such that for all $i \in [1,k]$ we have $|a_i|+|b_i|=|c_i|$; $a_i=c_i$ (hence $b_i=0$) if $c_i>0$; $a_i \le 0 \le b_i$ if $c_i<0$ (remove the 0 from the list $(a_1,b_1,a_2,b_2,\ldots,a_k,b_k)$). That is, $D$ is obtained from $C$ by *broken negative parts operations.*
- $C \xleftarrow{R} D$ if $C$ is finer that $D \,|\!\!\models n$, that is, $D$ can be obtained from $C$ by summing consecutive parts of $C$ having the same sign (*refinement operations*).

*Example 3.13* - Let $C=(1,\bar{2},\bar{1})$, then

$$\left\{D \,|\!\!\models 4 \,|\, C \xleftarrow{B} D\right\} = \{(1,\bar{2},\bar{1}),(1,\bar{1},1,\bar{1}),(1,\bar{1},1,1),(1,2,\bar{1}),(1,\bar{2},1),(1,2,1)\}.$$

*Remark 3.14* - Let $C,D \,|\!\!\models n$, then we have $C \leftarrow D$ if and only if $S_C \subset \mathcal{A}_D$. We deduce easily from definitions, Lemma 2.21 and Example 2.23 the following properties for any $i \in [1,k-1]$:

- if $\mathrm{sign}\, c_i = \mathrm{sign}\, c_{i+1}$, $C=(c_1,\ldots,c_i,c_{i+1},\ldots,c_k) \xleftarrow{R} (c_1,\ldots,c_i+c_{i+1},\ldots,c_k) = D$, and this means that $\mathcal{A}_D = \mathcal{A}_C \uplus \left\{s_{|c_1|+\cdots+|c_i|}\right\}$;

- if $c_i, c_{i+1} < 0$, then
$$C = (c_1, \dots, c_i, c_{i+1}, \dots, c_k) \xleftarrow{B} (c_1, \dots, c_i + 1, 1, c_{i+1}, \dots, c_k) = D$$
(remove the 0 from the list), and this means that $\mathcal{A}_D = \mathcal{A}_C \uplus \{t_{|c_1|+\cdots+|c_i|}\}$;
- if $c_i < 0$ and $c_{i+1} > 0$, then
$$C = (c_1, \dots, c_i, c_{i+1}, \dots, c_k) \xleftarrow{B} (c_1, \dots, c_i + 1, 1, c_{i+1}, \dots, c_k) = D$$
(remove the 0 from the list), and this means that $\mathcal{A}_D \uplus \{s_{|c_1|+\cdots+|c_i|}\} = \mathcal{A}_C \uplus \{t_{|c_1|+\cdots+|c_i|}\}$. Moreover, as $s_{|c_1|+\cdots+|c_i|} \notin S_C$, we have $S_C \subset \mathcal{A}_D$, that is, $C \leftarrow D$;
- finally, if $c_i < 0$, then $C \xleftarrow{B} (c_1, \dots, c_i + 1, 1, c_{i+1}, \dots, c_k) = D$ and
$$C \xleftarrow{B} (c_1, \dots, c_i + 2, 2, c_{i+1}, \dots, c_k) = D'$$
(remove the 0 from the list), and this means that $\mathcal{A}_{D'} = \mathcal{A}_D \uplus \{t_{|c_1|+\cdots+|c_i|}\}$. Hence $S_C \subset \mathcal{A}_D \subset \mathcal{A}_{D'}$.

In all these cases, we have $C \leftarrow D$.

**Theorem 3.15.** *Let $C, D \Vdash n$, then $C \leftarrow D$ if and only if there is $E \,||\!\!= n$ such that $C \xleftarrow{B} E \xleftarrow{R} D$. Moreover, $E$ is uniquely determined.*

*Proof.* Suppose that $E$ exists, then it is easy to check (using Remark 3.14 and induction) that $S_C \subset \mathcal{A}_E \subset \mathcal{A}_D$, which implies $C \leftarrow D$.

Now, suppose that $C \leftarrow D$. As $S_C \subset \mathcal{A}_D$, it is easy to construct a unique $E \,||\!\!= n$ such that $\mathcal{A}_E \cap T_n = \mathcal{A}_D \cap T_n$ and $C \xleftarrow{B} E$ (hence $C \leftarrow B$). It remains to show that $E \xleftarrow{R} D$, that is, to show that $\mathcal{A}_E \cap S_{\bar{n}} \subset \mathcal{A}_D \cap S_{\bar{n}}$. Let $s_j \in \mathcal{A}_E \cap S_{\bar{n}}$, then either $j \in [|c_1| + \cdots + |c_{i-1}| + 1, \dots, |c_1| + \cdots + |c_i| - 1]$ hence $s_j \in S_C \subset \mathcal{A}_D$; or $j = |c_1| + \cdots + |c_i|$ and $c_i < 0$ and $c_{i+1} > 0$ by definition. But refinement operations do not act on parts having not the same sign, that is, $s_j \in \mathcal{A}_D$. □

*Example 3.16* - Consider the signed composition $C = (1, \bar{2}, \bar{1})$ . Then we obtain from Theorem 3.15 and Example 3.13
$$\begin{aligned} X_{(1,\bar{2},\bar{1})} &= Y_{(1,\bar{2},\bar{1})} \cup Y_{(1,\bar{3})} \cup Y_{(1,\bar{1},1,\bar{1})} \cup Y_{(1,\bar{1},1,1)} \cup Y_{(1,\bar{1},2)} \cup Y_{(1,2,\bar{1})} \cup Y_{(3,\bar{1})} \\ &\quad \cup Y_{(1,\bar{2},1)} \cup Y_{(1,2,1)} \cup Y_{(3,1)} \cup Y_{(1,3)} \cup Y_{(4)}. \end{aligned}$$

## 4. Coplactic space

4.1. **Robinson-Schensted correspondence for $W_D$.** In [20], the author defined a bijection between $W_n$ and a certain set of bitableaux, which sounds like a Robinson-Schensted correspondence. Let us recall here some of his results. A *bitableau* is a pair $T = (T^+, T^-)$ of tableaux. The *shape* of $T$ is the bipartition $(\lambda^+, \lambda^-)$, where $\lambda^+$ is the shape of $T^+$ and $\lambda^-$ is the shape of $T^-$: it is denoted by $\operatorname{sh} T$. We note $|T| = |\operatorname{sh} T|$. The bitableau $T$ is said to be *standard* if the set of numbers in $T^+$ and $T^-$ is $[1, m]$, where $m = |T|$, and if the fillings of $T^+$ and $T^-$ are increasing in rows and in column.

Let $D \,||\!\!= n$. Write $D = (d_1, \dots, d_r)$ and denote by $\mathcal{SBT}(D)$ the set of $r$-uples $T = (T_1, \dots, T_r)$ of bitableaux $T_i = (T_i^+, T_i^-)$ such that $|T_i| = |d_i|$, $T_i^- = \emptyset$ if $d_i < 0$, $T_i^+$ and $T_i^-$ are standard and the fillings of $T_i^+$ and $T_i^-$ are exactly the numbers in $I_{D,+}^{(i)} = [|d_1| + \cdots + |d_{i-1}| + 1, |d_1| + \cdots + |d_i|]$. The *shape* of $T$,

denoted by $\operatorname{sh} T$, is the $r$-uple of bipartitions $(\operatorname{sh} T_1, \ldots, \operatorname{sh} T_r)$. If $T \in \mathcal{SBT}(D)$, then $\operatorname{sh} T \in \operatorname{Bip}(D)$. If $\lambda \in \operatorname{Bip}(D)$, we denote by $\mathcal{SBT}_\lambda^D$ the set of elements $T \in \mathcal{SBT}(D)$ such that $\operatorname{sh} T = \lambda$. In [20], the author defined a bijection (which we call *generalized Robinson-Schensted correspondence*)

$$\begin{array}{rcl} \pi_D : W_D & \longrightarrow & \{(P,Q) \in \mathcal{SBT}(D) \times \mathcal{SBT}(D) \mid \operatorname{sh} P = \operatorname{sh} Q\} \\ w & \longmapsto & (\boldsymbol{P}_D(w), \boldsymbol{Q}_D(w)). \end{array}$$

Note that, in [20] (see also [5, Section 3]), the bijection has been defined only for $D = (n)$. It is not difficult to deduce from this the bijection $\pi_D$ for general $D$. To this bijection is associated a partition of $W_n$ as follows: if $Q \in \mathcal{SBT}(D)$, we set

$$Z_Q^D = \{w \in W_D \mid \boldsymbol{Q}_D(w) = Q\}.$$

Then

$$W_D = \coprod_{Q \in \mathcal{SBT}(D)} Z_Q^D.$$

4.2. **Properties.** First, note that the bijection $\pi_D$ satisfies the following property: if $w \in W_n$, then

$$\pi_D(w^{-1}) = (\boldsymbol{Q}_D(w), \boldsymbol{P}_D(w)). \tag{4.1}$$

In particular, if $Q$ and $Q'$ are two elements of $\mathcal{SBT}(D)$, then

$$|Z_Q^D \cap (Z_{Q'}^D)^{-1}| = \begin{cases} 1 & \text{if } \operatorname{sh} Q = \operatorname{sh} Q', \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

*Remark.* $\pi_{\bar{n}}$ is the usual Robinson-Schensted correspondence. For simplification, we denote by $Z_Q = Z_Q^{(n)}$ if $Q \in \mathcal{SBT}(n)$.

In [5, Section 3], the authors give an another way to define the equivalence relation associated to this partition which looks like coplactic equivalence or dual-Knuth equivalence. If $w$, $w' \in W_D$, we write $w \smile_D w'$ if $w'w^{-1} \in S_{D^-} \subset S_{\bar{n}} = \{s_1, \ldots, s_{n-1}\}$ and $\mathcal{D}'_D(w^{-1}) \not\subset \mathcal{D}'_D(w'^{-1})$ and $\mathcal{D}'_D(w'^{-1}) \not\subset \mathcal{D}'_D(w^{-1})$. Note that the relation $\smile_D$ is symmetric. We denote by $\sim_D$ the reflexive and transitive closure of $\smile_D$. It is an equivalence relation, called the *coplactic equivalence relation*. The equivalence classes for this relation are called the *coplactic classes* of $W_D$. We denote by $\operatorname{Cop}(W_D)$ the set of coplactic classes for the relation $\sim_D$.

By [5, Proposition 3.8], we have, for every $w$, $w' \in W_D$,

$$w \sim_D w' \Longleftrightarrow \boldsymbol{Q}_D(w) = \boldsymbol{Q}_D(w'). \tag{4.3}$$

So $\operatorname{Cop}(W_D) = \{Z_Q^D \mid Q \in \mathcal{SBT}(D)\}$.

*Remark 4.4* - The relation $\smile_D$ has a useful combinatorial interpretation (see [5, Proof of Proposition 3.8]): $w \smile_D s_i w$ ($s_i \in S_{D^-}$) if and only if:

- either $\operatorname{sign}\big(w(i)\big) \neq \operatorname{sign}\big(w(i+1)\big)$;
- or $\operatorname{sign}\big(w(i)\big) = \operatorname{sign}\big(w(i+1)\big)$, then $s_i w$ is obtained from $w$ by a classical dual-Knuth transformation; that is, $w^{-1}(i-1)$ or $w^{-1}(i+2)$ lie between $w^{-1}(i)$ and $w^{-1}(i+1)$.

**Proposition 4.5.** *Let $w, w' \in W_D$, then $w \sim_D w' \Rightarrow \mathcal{D}'_D(w) = \mathcal{D}'_D(w')$.*

*Proof.* We may, and we will, assume that $D = (n)$. If $T_0$ is a Young tableau, let $\mathcal{D}(T_0) = \{s_p \in S_{\bar{n}} \mid p+1 \text{ lies in a row above the row containing } p\}$. Let ${}^tT_0$ denote the transposed tableau of $T$. If $T = (T^+, T^-)$ is a standard bitableau, we set

$$\mathcal{D}'(T) = \bigl\{t_p \,|\, p \in T^-\bigr\} \uplus \bigl\{s_p \,|\, p \in T^+ \text{ and } p+1 \in T^-\bigr\} \uplus \mathcal{D}(T^+) \uplus \mathcal{D}\bigl({}^tT^-\bigr).$$

Then it is easy to check that $\mathcal{D}'_n(w) = \mathcal{D}'(\boldsymbol{Q}(w))$. This completes the proof of the proposition. $\square$

*Example 4.6* - Let

$$T = \left( \begin{array}{|c|c|} \hline 1 & 7 \\ \hline 6 & 9 \\ \hline 8 \\ \cline{1-1} \end{array} \quad , \quad \begin{array}{|c|c|c|} \hline 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \right) \quad \in \mathcal{SBT}(9).$$

Then

$$\mathcal{D}'(T) \quad = \quad \{s_1, s_3, s_6, s_8, t_2, t_3, t_4, t_5\}.$$

*Remark 4.7* - Using Proposition 4.5, we may assign a signed composition $\boldsymbol{C}(Q) \mathrel{||}\!\!= n$ to any standard bitableau $Q \in \mathcal{SBT}(n)$ by setting $\boldsymbol{C}(Q) = \boldsymbol{C}(w)$ for any $w \in W_n$ such that $\boldsymbol{Q}(w) = Q$. One can determine $\boldsymbol{C}(Q)$ directly from $Q$ thanks to the following procedure, which is a combinatorial translation of the proof of Proposition 4.5. First one looks for maximal subwords $j\ j+1\ j+2 \dots k$ of $1\ 2\ 3 \dots n$ such that

- either the numbers $j,\ j+1,\ j+2,\ \dots,\ k$ can be found in this order in $Q^+$ when one goes from left to right (changes of rows are allowed)
- or they can be found in this order in $Q^-$ when one goes from top to bottom (changes of column are allowed).

The word $1\ 2\ 3 \dots n$ is then the concatenation of these maximal subwords, and the signed composition $\boldsymbol{C}(Q)$ is the sequence of the lengths of these subwords, adorned with a minus sign if the letters of the subword can be found in $Q^-$. As an example, consider $Q = (Q^+, Q^-)$ with

$$Q^+ = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 6 & 7 & 8 & 13 \\ \hline 9 & 11 & 12 \\ \cline{1-3} 10 \\ \cline{1-1} \end{array} \qquad \text{and} \qquad Q^- = \begin{array}{|c|c|} \hline 3 & 14 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 15 \\ \cline{1-1} \end{array}\ .$$

The partition of $1\ 2 \dots 14\ 15$ in maximal subwords is $1\ 2 \mid 3\ 4\ 5 \mid 6\ 7\ 8 \mid 9 \mid 10\ 11\ 12\ 13 \mid 14\ 15$, from what we can deduce that $\boldsymbol{C}(Q) = (2, \bar{3}, 3, 1, 4, \bar{2})$.

Therefore, we have by definitions

$$X_C = \coprod_{C \leftarrow \boldsymbol{C}(Q)} Z_Q.$$

**Proposition 4.8.** *Let $C, D \mathrel{||}\!\!= n$ such that $C \subset D$. Let $w,\ w' \in W_C$ and $x,\ x' \in X_C^D$, then:*

(a) *If $w \sim_C w'$, then $wx^{-1} \sim_D w'x^{-1}$.*
(b) *If $xw \sim_D x'w'$, then $w \sim_C w'$.*
(c) *If $w \sim_C w'$, then $w_C w \sim_C w_C w'$ and $ww_C \sim_C w'w_C$.*

*Proof.* (c) is clear. Let us now prove (a). We may assume that $w \smile_C w'$. But $\mathcal{D}_C(w^{-1}) \subset \mathcal{D}_D(xw^{-1})$ and $\mathcal{D}_C(w'^{-1}) \subset \mathcal{D}_D(xw'^{-1})$. So $wx^{-1} \smile_D w'x^{-1}$.

We now prove (b). If $W_C$ is a standard parabolic subgroup of $W_D$, and using the fact that coplactic classes are left cells for a particular choice of parameters [5, Theorem 7.7], then (b) follows from [8]. Therefore, by taking direct products and by arguing by induction on $|X_C^D|$, we may now assume that $D = (n)$ and $C = (k, l)$ with $k$, $l \geq 1$ and $k + l = n$.

Let us start by proving (a). We may assume that $w \smile_C w'$. But $\mathcal{D}_C(w^{-1}) \subset \mathcal{D}_D(xw^{-1})$ and $\mathcal{D}_C(w'^{-1}) \subset \mathcal{D}_D(xw'^{-1})$. So $wx^{-1} \smile_D w'x^{-1}$.

Let us now prove (b). We may assume that $xw \smile_D x'w'$. Let $Q = \boldsymbol{Q}_C(w) = \boldsymbol{Q}_C(w')$. From Remark 4.4, we have two cases: either $x'w'$ is obtained from $xw$ by a dual-Knuth relation, or $x'w' = s_i xw$ and $\operatorname{sign}(xw(i)) \neq \operatorname{sign}(xw(i+1))$. In the first case, observe that, for any $i \in [1, k-1]$ and $i \in [k+1, k+l-1]$, $xw(k) < xw(k+1)$ if and only if $w(k) < w(k+1)$, since $x \in X_{(k,l)} = X_{(\bar{k},\bar{l})}^{(\bar{n})} \subset W_{\bar{n}}$. Then we conclude by Remark 4.4 (which is exactly the result of Lascoux and Schützenberger on the shuffle of plactic classes [13]).

In the second case, observe that, for any $k \in [1, n]$,

$$(\star) \qquad \operatorname{sign}(w(k)) = \operatorname{sign}(xw(k)) = \operatorname{sign}(s_i xw(k))$$

since $X_{(k,l)} = X_{(\bar{k},\bar{l})}^{(\bar{n})} \subset W_{\bar{n}}$ and $s_i \in S_{\bar{n}}$. If $s_i x = x'$, then $w = w'$ and the result follows. If $s_i x = x s_j$, with $s_j \in S_{(\bar{k},\bar{l})}$ (by Deodhar's Lemma), then $x' = x$. Therefore $w' = s_j w$, by $(\star)$ and Remark 4.4. So $w \smile_C w'$. □

4.3. **Coplactic space.** Let $D \mathrel{||}\!\!= n$. If $Q \in \mathcal{SBT}(D)$, we set

$$z_Q^D = \sum_{w \in Z_Q^D} w \qquad \in \mathbb{Z}W_D.$$

Now, let

$$\mathcal{Q}_D = \bigoplus_{Q \in \mathcal{SBT}(D)} \mathbb{Z} z_Q^D \qquad \subset \mathbb{Z}W_D$$

and

$$\mathcal{Q}_D^{\perp} = \bigoplus_{\substack{Q, Q' \in \mathcal{SBT}(D) \\ \operatorname{sh} Q = \operatorname{sh} Q'}} \mathbb{Z}(z_Q^D - z_{Q'}^D) \qquad \subset \mathcal{Q}_D.$$

Then, by Proposition 4.5, we have

$$\textbf{(4.9)} \qquad \Sigma'(W_D) \subset \mathcal{Q}_D.$$

The next proposition justifies the notation $\mathcal{Q}_D^{\perp}$:

**Proposition 4.10.** $\mathcal{Q}_D^{\perp} = \{x \in \mathcal{Q}_D \mid \forall\, y \in \mathcal{Q}_D,\ \tau_D(xy) = 0\}$.

*Proof.* Let $\mathcal{Q}'_D = \{x \in \mathcal{Q}_D \mid \forall\, y \in \mathcal{Q}_D,\ \tau_D(xy) = 0\}$. Let $Q$ and $Q'$ be elements of $\mathcal{SBT}(D)$. Then, by 4.2, we have

$$\textbf{(4.11)} \qquad \tau_D(z_Q^D z_{Q'}^D) = \begin{cases} 1 & \text{if } \operatorname{sh} Q = \operatorname{sh} Q', \\ 0 & \text{otherwise.} \end{cases}$$

This shows in particular that $\mathcal{Q}_D^{\perp} \subset \mathcal{Q}'_D$.

Let us now prove that $\mathcal{Q}'_D \subset \mathcal{Q}_D^{\perp}$. Now, since $\mathcal{Q}_D/\mathcal{Q}_D^{\perp}$ is torsion free, it is sufficient to prove that $\dim_{\mathbb{Q}} \mathbb{Q}\mathcal{Q}'_D \leq \dim_{\mathbb{Q}} \mathbb{Q}\mathcal{Q}_D^{\perp}$. But, by construction, we have

$\dim_{\mathbb{Q}} \mathbb{Q}\mathcal{Q}_D - \dim_{\mathbb{Q}} \mathbb{Q}\mathcal{Q}_D^{\perp} = |\operatorname{Bip}(D)|$. Moreover, by Proposition 3.8, we have $\dim_{\mathbb{Q}} \mathbb{Q}\mathcal{Q}_D - \dim_{\mathbb{Q}} \mathbb{Q}\mathcal{Q}'_D \geq |\operatorname{Irr} W_D| = |\operatorname{Bip}(D)|$. □

The next lemma is a generalization to our case of a result of Blessenohl and Schocker concerning the symmetric group [3].

**Proposition 4.12.** *We have* $\mathcal{Q}_D = \Sigma'(W_D) + \mathcal{Q}_D^{\perp}$ *and* $\Sigma'(W_D) \cap \mathcal{Q}_D^{\perp} = \operatorname{Ker} \theta_D$.

*Proof.* Let us first prove that $\mathcal{Q}_D = \Sigma'(W_D) + \mathcal{Q}_D^{\perp}$. For this, we may, and we will, assume that $D = (n)$. We first need to introduce an order on bipartitions of $n$. We denote by $\leq_{\text{sl}}$ the lexicographic order on $\operatorname{Bip}(n)$ induced by the following order on $I_n$:

$$\bar{1} <_{\text{sl}} \bar{2} <_{\text{sl}} \cdots <_{\text{sl}} \bar{n} <_{\text{sl}} 1 <_{\text{sl}} 2 <_{\text{sl}} \cdots <_{\text{sl}} n.$$

If $\lambda$ is a bipartition of $n$, we denote by $Q_\lambda = Q(\eta_{\hat{\lambda}})$. If $\lambda = (\lambda^+, \lambda^-)$, then it is easy to check that $\operatorname{sh} Q_\lambda = (\lambda^+, {}^t\lambda^-) = \lambda^*$, where ${}^t\lambda^-$ is the transpose of the partition $\lambda$, and, using Remarque 4.7, that $Q_\lambda$ is obtained by numbered $Q_\lambda^+$ (resp. ${}^tQ_\lambda^-$) the first column first, then the second one and so on. Now, let $Q \in \mathcal{SBT}(n)$. Then:

**Lemma 4.13.** *Assume that* $Z_Q \subset X_{\hat{\lambda}}$*, then* $\lambda \leq_{\text{sl}} (\operatorname{sh} Q)^*$*. Moreover, if* $\operatorname{sh} Q = \lambda^*$*, then* $Q = Q_\lambda$.

*Proof.* First, we easily check (using Remark 4.7), that $\boldsymbol{\lambda}\big(\boldsymbol{C}(Q)\big) \leq_{\text{sl}} (\operatorname{sh} Q)^*$ with equality if and only if $Q = Q_\lambda$.

Then, observe (using Theorem 3.15), that $\lambda \leq_{\text{sl}} \boldsymbol{\lambda}\big(\boldsymbol{C}(Q)\big)$ with equality if and only if $\boldsymbol{C}(Q) = \hat{\lambda}$. This conclude the proof. □

We are now ready to prove by descending induction on $(\operatorname{sh} Q)^*$ that $z_Q \in \Sigma'(W_n) + \mathcal{Q}_n^{\perp}$. If $(\operatorname{sh} Q)^* = (n, \emptyset)$, then $Z_Q = \{1\} = Y_n$. So $z_Q = y_n \in \Sigma'(W_n)$.

Now, assume that $(\operatorname{sh} Q)^* <_{\text{sl}} (n, \emptyset)$ and that $z_{Q'} \in \Sigma'(W_n) + \mathcal{Q}_n^{\perp}$ for every $Q' \in \mathcal{SBT}(n)$ such that $(\operatorname{sh} Q)^* <_{\text{sl}} (\operatorname{sh} Q')^*$. Let $\lambda = (\operatorname{sh} Q)^*$. Then $z_Q = z_{Q_\lambda} + (z_Q - z_{Q_\lambda}) \in z_{Q_\lambda} + \mathcal{Q}_n^{\perp}$. So it is sufficient to prove that $z_{Q_\lambda} \in \Sigma'(W_n) + \mathcal{Q}_n^{\perp}$. But, by Lemma 4.13 $x_{\hat{\lambda}} - z_{Q_\lambda}$ is a sum of $z_{Q'}$ with $\lambda <_{\text{lex}} (\operatorname{sh} Q)^*$. Hence, by the induction hypothesis, we have $x_\lambda - z_{Q_{\hat{\lambda}}} \in \Sigma'(W_n) + \mathcal{Q}_n^{\perp}$, as desired.

Now, let us prove that $\Sigma'(W_D) \cap \mathcal{Q}_D^{\perp} = \operatorname{Ker} \theta_D$. The natural map $\Sigma'(W_D) \to \mathcal{Q}_D/\mathcal{Q}_D^{\perp}$ is surjective, so $\operatorname{rank}_{\mathbb{Z}} \Sigma'(W_D) \cap \mathcal{Q}_D^{\perp} = \operatorname{rank}_{\mathbb{Z}} \operatorname{Ker} \theta_D$. Since the $\mathbb{Z}$-modules $\Sigma'(W_D)/(\Sigma'(W_D) \cap \mathcal{Q}_D^{\perp})$ and $\Sigma'(W_D)/\operatorname{Ker} \theta_D$ are torsion free, it is sufficient to prove that $\Sigma'(W_D) \cap \mathcal{Q}_D^{\perp}$ is contained in $\operatorname{Ker} \theta_D$. But this follows from Proposition 4.10 and Corollary 3.9. □

Using Proposition 4.12, we can easily extend the linear map $\theta_D$ to a linear map $\tilde{\theta}_D : \mathcal{Q}_D \to \mathbb{Z} \operatorname{Irr} W_D$. If $x \in \mathcal{Q}_D$, write $x = a + b$ with $a \in \Sigma'(W_D)$ and $b \in \mathcal{Q}_D^{\perp}$ and set

$$\tilde{\theta}_D(x) = \theta_D(a).$$

Then Proposition 4.12 shows that $\tilde{\theta}_D$ is well-defined (that is, $\theta_D(a)$ does not depend on the choice of $a$ and $b$).

**Theorem 4.14.** *Let* $D \,|\!\!\models n$*. Then:*

(a) $\tilde{\theta}_D$ *is an extension of* $\theta_D$ *to* $\mathcal{Q}_D$*;*
(b) $\operatorname{Ker}\tilde{\theta}_D = \mathcal{Q}_D^{\perp}$*;*
(c) *if* $x$ *and* $y$ *are two elements of* $\mathcal{Q}_D$*, then* $\tau_D(xy) = \langle\, \tilde{\theta}_D(x), \tilde{\theta}_D(y)\,\rangle_D$*;*
(d) *the diagram*

$$\begin{tikzcd} \mathcal{Q}_D \arrow[r, "\tilde{\theta}_D"] \arrow[dr, "\mathrm{aug}_D"'] & \mathbb{Z}\operatorname{Irr} W_D \arrow[d, "\deg_D"] \\ & \mathbb{Z} \end{tikzcd}$$

*is commutative;*
(e) *if* $x \in \mathcal{Q}_D$*, then*

$$\tilde{\theta}_D(x) = |W_D| \sum_{\lambda \in \mathrm{Bip}(D)} \frac{\tau_D(xE_\lambda^D)}{|\mathcal{C}_\lambda^D|} f_\lambda^D.$$

*Proof.* (a) and (b) are easy. (c) follows from Proposition 3.8 and Proposition 4.10. (d) follows from the commutativity of the diagram 3.6 and from the fact that $\mathrm{aug}_D(\mathcal{Q}_D^{\perp}) = 0$ (indeed, if $Q$ and $Q'$ are two elements of $\mathcal{SBT}(D)$ of the same shape, then $|Z_Q^D| = |Z_{Q'}^D|$). Using Proposition 4.12, it is sufficient to prove (e) for $x \in \Sigma'(W_D)$ or $x \in \mathcal{Q}_D^{\perp}$. If $x \in \Sigma'(W_D)$, this follows from Proposition 3.10. If $x \in \mathcal{Q}_D^{\perp}$, this follows from Proposition 4.10. $\square$

*Remark.* In the theorem, the case $D = (\bar{n})$ is precisely the symmetric group case.

**Corollary 4.15.** *If* $\tilde{\theta}$ *is an extension of* $\theta_D$ *to the* $\mathcal{Q}_D$ *such that, for all* $x$ *and* $y$ *in* $\mathcal{Q}_D$*,* $\tau_D(xy) = \langle\, \tilde{\theta}(x), \tilde{\theta}(y)\,\rangle_D$*, then* $\tilde{\theta} = \tilde{\theta}_D$*.*

*Proof.* Assume that $\tilde{\theta}$ is an extension of $\theta_D$ to $\mathcal{Q}_D$ such that $\tau_D(xy) = \langle\, \tilde{\theta}(x), \tilde{\theta}(y)\,\rangle_D$ for all $x$ and $y$ in $\mathcal{Q}_D$. Then, if $x \in \mathcal{Q}_D^{\perp}$ and $\chi \in \mathbb{Z}\operatorname{Irr} W_D$, then there exists $y \in \mathcal{Q}_D$ such that $\tilde{\theta}_D(y) = \chi$. So

$$\langle\, \chi, \tilde{\theta}(x)\,\rangle_D = \langle\, \tilde{\theta}(y), \tilde{\theta}(x)\,\rangle_D = \tau_D(xy) = 0$$

by hypothesis and by Proposition 4.10. Since $\langle\, .,.\,\rangle_D$ is a perfect pairing on $\mathbb{Z}\operatorname{Irr} W_D$, we get that $\tilde{\theta}(x) = 0$. So $\tilde{\theta}$ coincides with $\tilde{\theta}_D$ on $\Sigma'(W_D)$ and on $\mathcal{Q}_D^{\perp}$, so $\tilde{\theta} = \tilde{\theta}_D$ by Proposition 4.12. $\square$

Let $\lambda \in \mathrm{Bip}(D)$. Let $Q \in \mathcal{SBT}(D)$ be of shape $\lambda$. Now, let

$$\xi_\lambda = \tilde{\theta}_D(z_Q).$$

Then $\xi_\lambda$ depends only on $\lambda$ and not on the choice of $Q$. Moreover, $\xi_\lambda \in \mathbb{Z}\operatorname{Irr} W_D$, $\deg_D \xi_\lambda = |Z_Q| > 0$ (see Theorem 4.14 (d)) and, by Theorem 4.14 (c) and 4.11, we have $\langle\, \xi_\lambda, \xi_\lambda\,\rangle_D = 1$. This shows that $\xi_\lambda \in \operatorname{Irr} W_D$. So we have proved the following proposition:

**Proposition 4.16.** *The map* $\mathrm{Bip}(D) \to \operatorname{Irr} W_D$*,* $\lambda \mapsto \xi_\lambda$ *is well-defined and bijective.*

*Remark 4.17* - If $T = (T^+, T^-) \in \mathcal{SBT}(n)$, we denote by $T^\vee$ the standard bitableau $(T^-, T^+)$. If $\lambda = (\lambda^+, \lambda^-) \in \mathrm{Bip}(n)$, we set $\lambda^\vee = (\lambda^-, \lambda^+) \in \mathrm{Bip}(n)$. In particular, $\mathrm{sh}\, T^\vee = (\mathrm{sh}\, T)^\vee$.

Now, let $w \in W_n$. Then $\pi_n(w_n w) = (\boldsymbol{P}(w)^\vee, \boldsymbol{Q}(w)^\vee)$. Therefore, if $Q \in \mathcal{SBT}(n)$ then $w_n Z_Q = Z_{Q^\vee}$. This shows in particular that $w_n \mathcal{Q}_n = \mathcal{Q}_n$ and that $w_n \mathcal{Q}_n^\perp = \mathcal{Q}_n^\perp$. Moreover,

**(4.18)** $$\tilde{\theta}_n(w_n z) = \varepsilon_n \tilde{\theta}_n(z)$$

for all $z \in \mathcal{Q}_n$. Indeed, this equality is true if $z \in \Sigma'(W_n)$ by Theorem 3.7 and it is obviously true if $z \in \mathcal{Q}_n^\perp$. So we can conclude using Proposition 4.12. In particular, if $\lambda \in \mathrm{Bip}(n)$, then

**(4.19)** $$\xi_{\lambda^\vee} = \varepsilon_n \xi_\lambda.$$

*Remark 4.20* - Let $Q \in \mathcal{SBT}(n)$ be such that $Q^- = \emptyset$. Then $z_Q \in \mathcal{Q}_{\bar{n}}$. Therefore, $\mathcal{Q}_{\bar{n}} \subset \mathcal{Q}_n$. Moreover, $\mathcal{Q}_{\bar{n}}^\perp = \mathcal{Q}_n^\perp \cap \mathcal{Q}_{\bar{n}}$. Therefore, it follows from the commutativity of Diagram 3.4 that the diagram

**(4.21)** $$\begin{tikzcd} \mathcal{Q}_{\bar{n}} \arrow[r, hook] \arrow[d, "\tilde{\theta}_{\bar{n}}"'] & \mathcal{Q}_n \arrow[d, "\tilde{\theta}_n"] \\ \mathbb{Z}\,\mathrm{Irr}\,\mathfrak{S}_n \arrow[r, hook, "p_n^*"] & \mathbb{Z}\,\mathrm{Irr}\,W_n \end{tikzcd}$$

is commutative. In particular, if $\lambda = (\lambda^+, \emptyset)$ is the shape of $Q$, and if we denote by $\chi_{\lambda^+}^{\bar{n}}$ the irreducible character of $\mathfrak{S}_n$ associated to $\lambda^+$ (apply Proposition 4.16 with $D = (\bar{n})$), we have

**(4.22)** $$\xi_\lambda = p_n^* \chi_{\lambda^+}^{\bar{n}}.$$

4.4. **Induction.** We first start by an easy consequence of Proposition 4.8.

**Lemma 4.23.** *Let $C$, $D \,||\!\!= n$ be such that $C \subset D$. Let $x \in \mathcal{Q}_C$. Then*

(a) $x_C^D x \in \mathcal{Q}_D$.

(b) *If $x \in \mathcal{Q}_C^\perp$, then $x_C^D x \in \mathcal{Q}_D^\perp$.*

*Proof.* (a) By linearity, we may assume that $x = z_Q^C$ with $Q \in \mathcal{SBT}(C)$. Then, by Proposition 4.8 (a), we have that $X_C^D.Z_Q^C$ is a union of coplactic classes. So $x_C^D x \in \mathcal{Q}_D$.

(b) By linearity, we may assume that $x = z_Q^C - z_{Q'}^C$ where $Q$, $Q' \in \mathcal{SBT}(C)$ and $\mathrm{sh}(Q) = \mathrm{sh}(Q')$. We denote by $\psi : Z_Q^C \to Z_{Q'}^C$ the unique bijection such that $\boldsymbol{P}_C(\psi(w)) = \boldsymbol{P}_C(w)$ for every $w \in Z_Q^C$.

Then $x_C^D x = x_C^D . \sum_{w \in Z_Q^C} (w - \psi(w))$. But, if $a \in X_C^D$ and $w \in Z_Q^C$, then $\boldsymbol{P}_D(aw) = \boldsymbol{P}_D(a\psi(w))$ by Proposition 4.8 (b) and 4.1. We set $\psi'(aw) = a\psi(w)$, then the map $\psi' : X_C^D.Z_Q^C \to X_C^D.Z_{Q'}^C$ is bijective and satisfies $\mathrm{sh}\,\boldsymbol{Q}_D(\psi'(w)) = \mathrm{sh}\,\boldsymbol{Q}_D(w)$ for every $w \in X_C^D.Z_Q^C$.

Now, let $\lambda \in \mathrm{Bip}(D)$ and let $\mathcal{E}_\lambda$ (resp. $\mathcal{E}'_\lambda$) be the set of $w \in X_C^D.Z_Q^C$ (resp. $w \in X_C^D.Z_{Q'}^C$) such that $\mathrm{sh}\,\boldsymbol{Q}_D(w) = \lambda$. Then $\psi'$ induces a bijection between

$\mathcal{E}_\lambda$ and $\mathcal{E}'_\lambda$. Write $\mathcal{E}_\lambda = \coprod_{i=1}^{r} Z_{Q_i}^D$ and $\mathcal{E}'_\lambda = \coprod_{i=1}^{r'} Z_{Q'_i}^D$, using (a). Then, since $|Z_{Q_1}^D| = \cdots = |Z_{Q_r}^D| = |Z_{Q'_1}^D| = \cdots = |Z_{Q_{r'}}^D|$ and $|\mathcal{E}_\lambda| = |\mathcal{E}'_\lambda|$, we have $r = r'$. This shows that $x_C^D x \in \mathcal{Q}_D^\perp$. □

**Corollary 4.24.** *Let $C$, $D \,|\!\!\models n$ be such that $C \subset D$. Then the diagram*

$$\begin{tikzcd}
\mathcal{Q}_C \arrow[r, "x_C^D\cdot"] \arrow[d, "\tilde{\theta}_C"'] & \mathcal{Q}_D \arrow[d, "\tilde{\theta}_D"] \\
\mathbb{Z}\operatorname{Irr} W_C \arrow[r, "\operatorname{Ind}_{W_C}^{W_D}"] & \mathbb{Z}\operatorname{Irr} W_D
\end{tikzcd}$$

*is commutative.*

*Proof.* This follows immediately from Proposition 4.23 and from the commutativity of the diagram 3.3. □

Now, if $\lambda \in \operatorname{Bip}(n)$, then we denote by $\chi_\lambda$ the irreducible character of $W_n$ associated to $\lambda$ via Clifford theory (see [9]). The link between the two parametrizations (the $\xi$'s and the $\chi$'s) is given by the following result:

**Corollary 4.25.** *If $\lambda$ is a bipartition of $n$, then $\xi_\lambda = \chi_{\lambda^*}$.*

*Proof.* Write $\lambda = (\lambda^+, \lambda^-)$, $k = |\lambda^+|$ and $l = |\lambda^-|$. Let $Q^+$ be a standard tableau of shape $\lambda^+$ filled with $\{l+1, l+2, \ldots, n\}$ and let $Q^-$ be a standard tableau of shape $\lambda^-$ filled with $\{1, 2, \ldots, l\}$. Then, by [5, Proposition 4.8],

$$Z_Q = X_{l,k}(w_l Z_{Q^-} \times Z_{Q^+}).$$

Therefore, by Corollary 4.24, we have

$$\xi_\lambda = \operatorname{Ind}_{W_{l,k}}^{W_n}(\tilde{\theta}_l(w_l Z_{Q^-}^l) \boxtimes \tilde{\theta}_k(Z_{Q^+}^k)).$$

So, by 4.22 and by Remark 4.17, we have

$$\xi_\lambda = \operatorname{Ind}_{W_{k,l}}^{W_n}\Big(p_k^* \chi_{\lambda^+}^{\bar{k}} \boxtimes \varepsilon_l(p_l^* \chi_{\lambda^-}^{\bar{l}})\Big).$$

The result now follows from [9]. □

## 5. Related Hopf algebras

5.1. **Hopf algebra of signed permutations.** Consider the graded $\mathbb{Z}$-module

$$\mathcal{SP} = \mathop{\oplus}_{n \geq 0} \mathbb{Z} W_n,$$

where $W_0 = 1$. In [1], Aguiar and Mahajan have shown that $\mathcal{SP}$ has a structure of Hopf algebra which is similar to the structure of the Malvenuto-Reutenauer Hopf algebra on permutations [15]. Moreover, they have shown that

$$\Sigma' = \mathop{\oplus}_{n \geq 0} \Sigma'(W_n)$$

is a Hopf subalgebra of $\mathcal{SP}$. We revise here the definition of the product and the coproduct on $\mathcal{SP}$ with our point of view.

Notation - If $C$ is a signed composition, then we denote by $x_C$ the element of $\mathcal{SP}$ lying in $\mathbb{Z}W_{|C|}$ corresponding to the $x_C$ defined in §3. Similarly, if $Q$ is a standard bitableau, then $z_Q \in \mathbb{Z}W_{|\,\mathrm{sh}\,Q|}$ is viewed as an element of $\mathcal{SP}$.

Let $(u,v) \in W_n \times W_m$, we denote $u \times v$ the corresponding element of $W_{n,m} \simeq W_n \times W_m$. If $w \in Wn, m$, we denote by $(w'_{(n)}, w''_{(m)})$ the corresponding element of $W_n \times W_m$. We now define

$$u * v = x_{n,m}(u \times v) \in \mathbb{Z}W_{n+m}.$$

We extend $*$ by linearity to a bilinear map $\mathcal{SP} \times \mathcal{SP} \to \mathcal{SP}$.

Now, let $w \in W_n$. Then, for each $i \in [0,n]$, we denote by $\pi_i(w)$ the unique element of $W_{i,n-i}$ such that $w \in \pi_i(w) X_{i,n-i}^{-1}$. We set

$$\Delta(w) = \sum_{i=0}^{n} \pi_i(w)'_{(i)} \otimes_{\mathbb{Z}} \pi_i(w)''_{(n-i)} \in \mathcal{SP} \otimes \mathcal{SP}.$$

We extend $\Delta$ by linearity to a map $\Delta : \mathcal{SP} \to \mathcal{SP} \otimes_{\mathbb{Z}} \mathcal{SP}$.

*Remark 5.1* - Combinatorially, we see this product as follows: let $w = w_1 \dots w_n$ be a word of length $n$ in the alphabet $I_n$, the *standardsigned permutation* is the unique element $\mathrm{sts}\,(w) \in W_n$ such that

$$\begin{cases} \mathrm{sts}\,(w)(i) < \mathrm{sts}\,(w)(j) \Leftrightarrow \big(w_i < w_j\big) \quad \text{or} \quad \big(w_i = w_j \text{ and } i < j\big) \\ \text{and} \quad \mathrm{sign}\,\big(\mathrm{sts}\,(w)(i)\big) = \mathrm{sign}\,(w_i). \end{cases}$$

Then

$$u{*}v = \sum_{w,w'} ww'$$

where $ww'$ is the concatenation of $w$ and $w'$; and the sum is taken over all words $w, w'$ on the alphabet $I_n$ such that $\mathrm{sts}\,(w) = u$, $\mathrm{sts}\,(w') = v$ and $\mathrm{alph}\,(u) \uplus \mathrm{alph}\,(v) = [1,n]$ (where $\mathrm{alph}\,(u) =$ the set of absolute values of the letters in $u$). For instance, $\bar{1}2 \times 2\bar{1} = \bar{1}24\bar{3}$ and

$$\begin{aligned} x_{(2,2)} &= y_{(2,2)} + y_{(4)} \\ &= 1234 + 1324 + 1423 + 2314 + 2413 + 3412. \end{aligned}$$

Hence $\bar{1}2{*}2\bar{1} = \bar{1}24\bar{3} + \bar{1}34\bar{2} + \bar{1}43\bar{2} + \bar{2}34\bar{1} + \bar{2}43\bar{1} + \bar{3}42\bar{1}$.

*Remark 5.2* - For $w \in W_n$ seen as a word on the alphabet $I_n$ and $i < j \in [1,n]$, we denote $w|[i,j]$ the subword obtained by taking only the digits such that their absolute values are in $[i,j]$. Then we see combinatorially the coproduct as

$$\Delta(w) = \sum_{i=0}^{n} w|[1,i] \otimes \mathrm{sts}\,\big(w|[i+1,n]\big).$$

As example, consider $w = \bar{2}31\bar{4}$, then we have the following decompositions:

$$w^{-1} = 3\bar{1}2\bar{4} = 3124(1 \times \bar{1}2\bar{3}) = 1324(2\bar{1} \times 1\bar{2}) = 1234(3\bar{1}2 \times \bar{1}).$$

Hence

$$w = \bar{2}31\bar{4} = (1 \times \bar{1}2\bar{3})2314 = (\bar{2}1 \times 1\bar{2})1324 = (\bar{2}31 \times \bar{1})1234.$$

Thus

$$\Delta(\bar{2}31\bar{4}) = \emptyset \otimes \bar{2}31\bar{4} + 1 \otimes \bar{1}2\bar{3} + \bar{2}1 \otimes 1\bar{2} + \bar{2}31 \otimes \bar{1} + \bar{2}31\bar{4} \otimes \emptyset.$$

*Example 5.3* - Let $C$ and $D$ be two signed composition. We denote by $C \sqcup D$ the signed composition obtained by concatenation of $C$ and $D$. Then

$$x_C * x_D = x_{C \sqcup D}.$$

*Example 5.4* - We have

$$\Delta(x_n) = \sum_{i=0}^{n} x_i \otimes_{\mathbb{Z}} x_{n-i}$$

and

$$\Delta(x_{\bar{n}}) = \sum_{i=0}^{n} x_{\bar{i}} \otimes_{\mathbb{Z}} x_{\overline{n-i}}.$$

We state here a result of Aguiar and Mahajan [1], with our basis consisting of the $x_C$.

**Theorem 5.5.** *The graded vector space $\mathcal{SP}$, with the product $*$ and the coproduct $\Delta$ is a connected graded Hopf algebra; and $\Sigma'$ is a Hopf subalgebra of $\mathcal{SP}$ which is freely generated by elements $(x_n)_{n \in \mathbb{Z} \setminus \{0\}}$ as algebra.*

If $x$, $y \in \mathcal{SP}$, we define the product $xy \in \mathcal{SP}$ as follows: if $x \in \mathbb{Z}W_n$ and $y \in \mathbb{Z}W_m$, then $xy = 0$ if $m \neq n$ and $xy$ coincides with the usual product $xy$ in $\mathbb{Z}W_n$ if $m = n$. Let $\tau : \mathcal{SP} \to \mathbb{Z}$ be the unique linear map which coincides with $\tau_n$ on $\mathbb{Z}W_n$. The the map $\mathcal{SP} \times \mathcal{SP} \to \mathbb{Z}$, $(x, y) \mapsto \tau(xy)$ is a scalar product on $\mathcal{SP}$. If $x$, $y \in \mathcal{SP}$, we set

$$\tau_{\otimes}(x \otimes y) = \tau(x)\tau(y).$$

The following proposition is easily checked from definitions:

**Proposition 5.6.** *$\mathcal{SP}$ is self-dual for $\tau$, that is,*

$$\tau_{\otimes}\big((u \otimes v)\Delta(w)\big) = \tau\big((u * v)w\big)$$

*for all $u$, $v$, $w \in \mathcal{SP}$.*

5.2. **The Hopf algebra of characters.** We give here a short recall of a result of Geissinger [10]. Consider the graded $\mathbb{Z}$-module

$$\mathcal{CHAR} = \bigoplus_{n \geq 0} \mathbb{Z} \operatorname{Irr} W_n.$$

If $k$ and $l$ are two natural numbers, we denote by $\iota_{k,l}$ the canonical isomorphism

$$\iota_{k,l} : \mathbb{Z} \operatorname{Irr} W_k \otimes_{\mathbb{Z}} \mathbb{Z} \operatorname{Irr} W_l \xrightarrow{\sim} \mathbb{Z} \operatorname{Irr} W_{k,l}.$$

Let $(\chi, \psi) \in \mathbb{Z} \operatorname{Irr} W_k \times \mathbb{Z} \operatorname{Irr} W_l$. We define

$$\chi \bullet \psi = \operatorname{Ind}_{W_{k,l}}^{W_{k+l}} \iota_{k,l}\big(\chi \otimes_{\mathbb{Z}} \psi\big) \quad \in \mathbb{Z} \operatorname{Irr} W_{k+l}.$$

Now, let $\chi \in \mathcal{CL}_{\mathbb{Q}} W_n$. We define

$$\operatorname{Res}(\chi) = \sum_{i=0}^{n} \iota_{i,n-i}^{-1} \operatorname{Res}_{W_{i,n-i}}^{W_n} \chi \quad \in \bigoplus_{i=0}^{n} \mathbb{Z} \operatorname{Irr} W_i \otimes_{\mathbb{Z}} \mathbb{Z} \operatorname{Irr} W_{n-i} \subset \mathcal{CHAR} \otimes_{\mathbb{Z}} \mathcal{CHAR}.$$

We denote $\langle \cdot, \cdot \rangle$ the unique scalar product on $\mathcal{CHAR}$ which coincides with $\langle \cdot, \cdot \rangle_n$ on $\mathbb{Z} \operatorname{Irr} W_n$ and such that $\mathbb{Z} \operatorname{Irr} W_n$ and $\mathbb{Z} \operatorname{Irr} W_m$ are orthogonal if $m \neq n$. We now define $\langle \cdot, \cdot \rangle_{\otimes}$ on $\mathcal{CHAR} \otimes_{\mathbb{Z}} \mathcal{CHAR}$ as follows: if $\chi$, $\chi'$, $\psi$, $\psi' \in \mathcal{CHAR}$, we set

$$\langle \chi \otimes \psi, \chi' \otimes \psi' \rangle_{\otimes} = \langle \chi, \chi' \rangle \langle \psi, \psi' \rangle.$$

Geissinger [10] has shown that $\mathcal{CHAR}$ with product $\bullet$ and coproduct Res is a connected graded Hopf algebra. Moreover, for any $\chi, \psi, \zeta \in \mathcal{CHAR}$, the reciprocity law of Frobenius can be viewed as

**(5.7)** $$\langle\, \chi \otimes \psi, \operatorname{Res} \zeta \,\rangle_\otimes = \langle\, \chi \bullet \psi, \zeta \,\rangle.$$

5.3. **The coplactic algebra and an Hopf epimorphism.** Let us intoduce

$$\mathcal{Q} = \mathop{\oplus}_{n\geq 0} \mathcal{Q}_n.$$

and

$$\mathcal{CHAR} = \mathop{\oplus}_{n\geq 0} \mathbb{Z} \operatorname{Irr} W_n.$$

We define $\theta : \Sigma' \to \mathcal{CHAR}$ and $\tilde{\theta} : \mathcal{Q} \to \mathcal{CHAR}$ by

$$\theta = \mathop{\oplus}_{n\geq 0} \theta_n \qquad \text{and} \qquad \tilde{\theta} = \mathop{\oplus}_{n\geq 0} \tilde{\theta}_n.$$

The first part of the following theorem shows that $\mathcal{Q}$ is a generalization of the Poirier-Reutenauer Hopf algebra of tableaux [17] to our case (see also [3]), and the second part shows that Jöllenbeck's construction generalizes to our case.

**Theorem 5.8.** *$\mathcal{Q}$ is a Hopf subalgebra of $\mathcal{SP}$ containing $\Sigma'$. Moreover, $\theta : \Sigma' \to \mathcal{CHAR}$ and $\tilde{\theta} : \mathcal{Q} \to \mathcal{CHAR}$ are surjective Hopf algebra homomorphisms.*

*Proof.* The fact that $\mathcal{Q}$ is a subalgebra of $\mathcal{SP}$ follows from Proposition 4.23. To prove that it is a subcoalgebra, we proceed as in the proof of the result of Poirier and Reutenauer [17], using Remark 4.4: let $Z$ be a coplactic class in $W_n$, $i \in [0, n]$ and $w \in Z$. Write $w = \pi_i(w)x$, where $x^{-1} \in X_{i,n-i}$. Let $u \in W_i$ such that $u \smile_{(i)} \pi_i(w)'_{(i)}$. As $\operatorname{sign}\big(x^{-1}w^{-1}(k)\big) = \operatorname{sign}\big(w^{-1}(k)\big)$ and $x^{-1}(l) < x^{-1}(l+1)$, for all $l \in [1, i-1]$ and for all $l \in [i+1, n-1]$, we easily check that $(u \times \pi_i(w)''_{(n-i)})x \smile_{(n)} w$, using Remark 4.4. Let $v \in W_{n-i}$ such that $v \smile_{(n-i)} \pi_i(w)''_{(n-i)}$, then $(u \times v)x \smile_{(n)} w$ as above. Therefore

$$\Delta\Big(\sum_{w\in Z} w\Big) = \sum_{i=0}^{n} \sum_{Z_i, Z_{n-i}} \Big(\sum_{u\in Z_i} u\Big) \otimes \Big(\sum_{v\in Z_{n-i}} v\Big),$$

where $Z_i$ (resp. $Z_{n-i}$) are coplactic classes in $W_i$ (resp. $W_{n-i}$).

We now need to prove that $\tilde{\theta}$ is an homomorphism of Hopf algebras. We first need a lemma concerning the symmetric bilinear form $\beta : (\mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{Q}) \times (\mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{Q}) \to \mathbb{Z}$, $(x, y) \mapsto \tau_\otimes(xy)$. Let $\tilde{\theta}_\otimes = \tilde{\theta} \otimes_{\mathbb{Z}} \tilde{\theta} : \mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{Q} \to \mathcal{CHAR} \otimes_{\mathbb{Z}} \mathcal{CHAR}$. Then:

**Lemma 5.9.** *$\operatorname{Ker} \tilde{\theta}_\otimes = \mathcal{Q} \otimes_{\mathbb{Z}} \operatorname{Ker} \tilde{\theta} + \operatorname{Ker} \tilde{\theta} \otimes_{\mathbb{Z}} \mathcal{Q}$ is the kernel of $\beta$.*

*Proof.* By Theorem4.14 (c), we have

$$\beta(x, y) = \langle\, \tilde{\theta}_\otimes(x), \tilde{\theta}_\otimes(y) \,\rangle_\otimes$$

for all $x$, $y \in \mathcal{Q} \otimes_{\mathbb{Z}} \mathcal{Q}$. This proves the lemma. $\square$

Proposition 5.6 and Lemma 5.9 show that $\operatorname{Ker}\tilde{\theta}$ is an ideal and a coideal of $\mathcal{Q}$. Since $\mathcal{Q} = \Sigma' + \operatorname{Ker}\tilde{\theta}$, it is sufficient to prove that $\theta$ is a bialgebra homomorphism. First, it is clear that $\tilde{\theta}(x_C * x_D) = \tilde{(}x_{C\sqcup D})$. So $\tilde{\theta}$ is an algebra homomorphism. Using this last propoerty and Theorem 5.5, it is sufficient to prove that $\tilde{\theta}_{\otimes}(\Delta(x_n)) = \operatorname{Res}(\tilde{\theta}(x_n))$ and $\tilde{\theta}_{\otimes}(\Delta(x_{\bar{n}})) = \operatorname{Res}(\tilde{\theta}(x_{\bar{n}}))$ But this follows easily from Example 5.4. $\square$

## 6. The case $n = 2$

In this Section, we will give a complete description of the algebra $\Sigma'(W_2)$. For simplification, we set $s = s_1$. Note that $t_1 = t$ and $t_2 = sts$. In other words, $S'_2 = \{s, t, sts\}$. Table I gives the correspondence between reduced decomposition of elements of $W_2$ and permutations of $I_2$ (if $w \in W_2$, we only give the couple $(w(1), w(2))$ since it determines $w$ as a permutation of $I_2$). It also gives the value of $\mathcal{U}'_2(w)$ and $\boldsymbol{C}(w)$. Table II gives representatives of the conjugacy classes of $W_2$. Table III gives, for each signed composition $C$ of 2, the subgroup $W_C$ of $W_2$, the set $S_C$, the elements $x_C$ and $y_C$ of $\mathbb{Z}W_2$ and also gives the value of $\mathcal{A}_C$. Table IV provides the decomposition of the induced characters $\operatorname{Ind}_{W_{\hat{\lambda}}}^{W_2} 1_{\hat{\lambda}} = \theta_2(x_{\hat{\lambda}})$ as a combination of the $\xi_\mu$, for $\lambda \Vdash 2$. Table V gives the character table of $\mathbb{Q}\Sigma'(W_2)$ (see Subsection 3.5). We give in Table VI a complete set of orthogonal primitive idempotents of $\mathbb{Q}\Sigma'(W_2)$. Table VII gives the Cartan matrix of $\Sigma'(W_2)$. As usual, the dots in the tables represent the number 0. Note that

$$w_2 = stst = tsts.$$

We conclude the Section by a description of the algebra $\mathbb{Q}\Sigma'(W_2)$ as a product of classical indecomposable algebras.

*Convention.* For avoiding the use of too many parenthesis, we have denoted by $\xi_{\hat{\lambda}}$, $\pi_{\hat{\lambda}}$ or $E_{\hat{\lambda}}$ the objects $\xi_\lambda$, $\pi_\lambda$ or $E_\lambda$ respectively. For instance, $\xi_{1,\bar{1}} = \xi_{((1),(1))}$ and $\pi_2 = \pi_{((2),\emptyset)}$ and $E_{\bar{1},\bar{1}} = E_{(\emptyset,(1,1))}$.

| $w$ | $(w(1), w(2))$ | $\mathcal{U}'_2(w)$ | $\boldsymbol{C}(w)$ |
|---|---|---|---|
| $1$ | $(1,2)$ | $\{s,t,sts\}$ | $(2)$ |
| $s$ | $(2,1)$ | $\{t,sts\}$ | $(1,1)$ |
| $t$ | $(\bar{1},2)$ | $\{s,sts\}$ | $(\bar{1},1)$ |
| $st$ | $(\bar{2},1)$ | $\{s,sts\}$ | $(\bar{1},1)$ |
| $ts$ | $(2,\bar{1})$ | $\{t\}$ | $(1,\bar{1})$ |
| $sts$ | $(1,\bar{2})$ | $\{t\}$ | $(1,\bar{1})$ |
| $tst$ | $(\bar{2},\bar{1})$ | $\{s\}$ | $(\bar{2})$ |
| $w_2$ | $(\bar{1},\bar{2})$ | $\emptyset$ | $(\bar{1},\bar{1})$ |

**Table I. Elements**

| $\hat{\lambda}$ | $c_\lambda$ | $\lvert\mathcal{C}_\lambda\rvert$ |
|---|---|---|
| $(2)$ | $st$ | 2 |
| $(1,1)$ | $w_2$ | 1 |
| $(1,\bar{1})$ | $t$ | 2 |
| $(\bar{2})$ | $s$ | 2 |
| $(\bar{1},\bar{1})$ | $1$ | 1 |

**Table II. Conjugacy classes**

| $C$ | $W_C$ | $S_C$ | $x_C$ | $y_C$ | $\mathcal{A}_C$ |
|---|---|---|---|---|---|
| $(2)$ | $W_2$ | $\{s,t\}$ | $1$ | $1$ | $\{s,t,sts\}$ |
| $(1,1)$ | $W_1 \times W_1$ | $\{t,sts\}$ | $1+s$ | $s$ | $\{t,sts\}$ |
| $(\bar{1},1)$ | $\mathfrak{S}_1 \times W_1$ | $\{sts\}$ | $1+s+t+st$ | $t+st$ | $\{t\}$ |
| $(1,\bar{1})$ | $W_1 \times \mathfrak{S}_1$ | $\{t\}$ | $1+s+ts+sts$ | $ts+sts$ | $\{s,sts\}$ |
| $(\bar{2})$ | $\mathfrak{S}_2$ | $\{s\}$ | $1+t+st+tst$ | $tst$ | $\{s,sts\}$ |
| $(\bar{1},\bar{1})$ | $1$ | $\emptyset$ | $\sum_{w\in W_2} w$ | $w_2$ | $\emptyset$ |

**Table III. Bases of $\Sigma'(W_2)$**

| | $\xi_2$ | $\xi_{1,1}$ | $\xi_{1,\bar{1}}$ | $\xi_{\bar{2}}$ | $\xi_{\bar{1},\bar{1}}$ |
|---|---|---|---|---|---|
| $\theta_2(x_2)$ | 1 | . | . | . | . |
| $\theta_2(x_{1,1})$ | 1 | 1 | . | . | . |
| $\theta_2(x_{1,\bar{1}})$ | 1 | 1 | 1 | . | . |
| $\theta_2(x_{\bar{2}})$ | 1 | . | 1 | 1 | . |
| $\theta_2(x_{\bar{1},\bar{1}})$ | 1 | 1 | 2 | 1 | 1 |

**Table IV. Decomposition of induced characters**

| | $x_2$ | $x_{1,1}$ | $x_{1,\bar{1}}$ | $x_{\bar{2}}$ | $x_{\bar{1},\bar{1}}$ |
|---|---|---|---|---|---|
| $\pi_2$ | 1 | . | . | . | . |
| $\pi_{1,1}$ | 1 | 2 | . | . | . |
| $\pi_{1,\bar{1}}$ | 1 | 2 | 2 | . | . |
| $\pi_{\bar{2}}$ | 1 | . | . | 2 | . |
| $\pi_{\bar{1},\bar{1}}$ | 1 | 2 | 4 | 4 | 8 |

**Table V. Character table of $\Sigma'(W_2)$**

*Remark.* Using these tables, one can check that $\theta_2(x_{\bar{2}})(x_{1,1}) = 6 \neq 4 = \theta_2(x_{1,1})(x_{\bar{2}})$. In other words, the symmetry property (see [4]) does not hold in our case.

$$
\begin{aligned}
E_2 &= x_2 - \frac{1}{2}x_{\bar{2}} - \frac{1}{4}x_{1,\bar{1}} + \frac{1}{4}x_{\bar{1},1} - \frac{1}{2}x_{1,1} + \frac{1}{4}x_{\bar{1},\bar{1}} \\
E_{1,1} &= \frac{1}{2}\left(x_{1,1} - \frac{1}{2}x_{1,\bar{1}} - \frac{1}{2}x_{\bar{1},1} + \frac{1}{4}x_{\bar{1},\bar{1}}\right) \\
E_{1,\bar{1}} &= \frac{1}{2}\left(x_{1,\bar{1}} - \frac{1}{2}x_{\bar{1},\bar{1}}\right) \\
E_{\bar{2}} &= \frac{1}{2}\left(x_{\bar{2}} - \frac{1}{2}x_{\bar{1},\bar{1}}\right) \\
E_{\bar{1},\bar{1}} &= \frac{1}{8}x_{\bar{1},\bar{1}}
\end{aligned}
$$

**Table VI. A complete set of orthogonal primitive idempotents**

We will now give the Cartan matrix of $\Sigma'(W_2)$. If $\mu \in \mathrm{Bip}(2)$, we denote by $\Pi_\mu$ the character of the projective cover $\mathbb{Q}\Sigma'(W_2)E_\mu$ of $\mathbb{Q}_\mu$. Write

$$\Pi_\mu = \sum_{\mu \in \mathrm{Bip}(2)} \gamma_{\lambda\mu}\pi_\lambda.$$

Then $(\gamma_{\lambda\mu})_{\lambda,\mu\in\mathrm{Bip}(2)}$ is the Cartan matrix of $\Sigma'(W_2)$. It is given in the following table:

| $\hat{\lambda} \setminus \hat{\mu}$ | $(2)$ | $(1,1)$ | $(1,\bar{1})$ | $(\bar{2})$ | $(\bar{1},\bar{1})$ |
|---|---|---|---|---|---|
| $(2)$ | 1 | . | . | . | . |
| $(1,1)$ | . | 1 | . | . | . |
| $(1,\bar{1})$ | . | . | 1 | 1 | . |
| $(\bar{2})$ | . | . | . | 1 | . |
| $(\bar{1},\bar{1})$ | . | . | . | . | 1 |

**Table VII. Cartan matrix of $\Sigma'(W_2)$**

Let $E_0 = E_{1,\bar{1}} + E_{\bar{2}}$. Then $(E_2, E_{1,1}, E_0, E_{\bar{2}})$ is a complete set of central primitive idempotents (they are of course orthogonal). Therefore, write $A_\omega = \mathbb{Q}\Sigma'(W_2)E_\omega$, for $\omega \in \{2, (1,1), 0, \bar{2}\}$. Then

$$\mathbb{Q}\Sigma'(W_2) = A_2 \oplus A_{1,1} \oplus A_{\bar{2}} \oplus A_0,$$

as a sum of algebras. Morever, $A_2 \simeq \mathbb{Q}$, $A_{1,1} \simeq \mathbb{Q}$, $A_{\bar{2}} \simeq \mathbb{Q}$. On the other hand,

$$A_0 = \mathbb{Q}E_{1,\bar{1}} \oplus \mathbb{Q}E_{\bar{2}} \oplus \mathbb{Q}(x_{1,\bar{1}} - x_{\bar{1},1}),$$

as a vector space. Now, let $B$ be the algebra

$$B = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Q}\}.$$

Then the $\mathbb{Q}$-linear map $\sigma : A_0 \to B$ such that

$$\sigma(E_{1,\bar{1}}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma(E_{\bar{2}}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma(x_{1,\bar{1}} - x_{\bar{1},1}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is an isomorphism of algebras. Therefore, we have an isomorphism of algebras

$$\mathbb{Q}\Sigma'(W_2) \simeq \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus B.$$

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# Appendix: Comparison with Specht's construction

Pierre Baumann and Christophe Hohlweg

The present text is an appendix to the article *Generalized descent algebra and construction of irreducible characters of hyperoctahedral groups*, by Cédric Bonnafé and the second present author. Our aim here is to relate two constructions of the irreducible characters of the hyperoctahedral groups: the one given in that article, and Specht's one [19]. Meant as a sequel to Bonnafé and Hohlweg's article, this text uses the same notations and references.

We first recall briefly Specht's construction, using Macdonald's book as a reference [14, I, Appendix B].

**Specht's construction.** Let $G$ be a finite group, let $G_*$ be the set of conjugacy classes in $G$ and let $G^*$ be the set of irreducible characters of $G$. Given a conjugacy class $c \in G_*$, we denote by $\zeta_c$ the order of the centralizer of an element of $c$. We denote the value of a character $\gamma$ of $G$ at any element of a comjugacy class $c \in G_*$ by $\gamma(c)$.

We denote the wreath product $G \wr \mathfrak{S}_n$ by $G_n$. This wreath product is the semidirect product $G^n \rtimes \mathfrak{S}_n$ for the action of $\mathfrak{S}_n$ on $G^n$ given by

$$\sigma \cdot (g_1, \dots, g_n) = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(n)}),$$

where $\sigma \in \mathfrak{S}_n$ and $(g_1, \dots, g_n) \in G^n$, so that we can always represent an element in $G_n$ as a product $(g_1 \dots, g_n)\, \sigma$.

Given a complex representation $V$ of $G$, we construct a complex representation $\eta_n(V)$ of $G_n$ on the space $V^{\otimes n}$ by letting a product $(g_1 \dots, g_n)\, \sigma$ acting on a pure tensor $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ in the following way:

$$((g_1 \dots, g_n)\, \sigma) \cdot (v_1 \otimes \cdots \otimes v_n) = (g_1 \cdot v_{\sigma^{-1}(1)}) \otimes \cdots \otimes (g_n \cdot v_{\sigma^{-1}(n)}).$$

The character of $\eta_n(V)$ does not depend on $V$ but only of its character; if $\rho$ denotes the latter, then we will denote the former by $\eta_n(\rho)$.

We let $\mathcal{P}$ be the set of all partitions, and we set $\mathcal{P}_G = \mathcal{P}^{G^*}$. Given an element $\lambda = (\lambda_\gamma)_{\gamma \in G^*}$ in $\mathcal{P}_G$, we denote by $|\lambda|$ the sum $\sum_\gamma |\lambda_\gamma|$.

Now let $\Lambda_\mathbb{C}$ be the (free) ring of symmetric polynomials with complex coefficients. As is well-known, $\Lambda_\mathbb{C}$ is generated over $\mathbb{C}$ by a countable family of algebraically independant elements: one can choose for generators the family $(h_n)_{n\geq 1}$ of complete symmetric functions or the family $(p_n)_{n\geq 1}$ of power sums. On the other hand, the family of Schur functions $(s_\lambda)_{\lambda\in\mathcal{P}}$ is a basis of the vector space $\Lambda_\mathbb{C}$. Following Macdonald, we denote by $\Lambda_\mathbb{C}(G)$ the ring of polynomials over $\mathbb{C}$ in the family of variables $\big(p_n(c)\big)_{n\geq 1, c\in G_*}$. Setting

$$p_n(\gamma) = \sum_{c \in G_*} \zeta_c^{-1} \gamma(c) p_r(c)$$

for any $\gamma \in G^*$, one can easily check that $\Lambda_\mathbb{C}(G)$ is also the ring of polynomials in the variables $\big(p_n(\gamma)\big)_{n\geq 1, \gamma \in G^*}$. Every symmetric polynomial $P \in \Lambda_\mathbb{C}$ can be expressed as a polynomial with complex coefficients in the power sums $p_n$; given $\gamma \in G^*$, we denote by $P(\gamma)$ the element of $\Lambda_\mathbb{C}(G)$ obtained by replacing the variables $p_n$ by the

variables $p_n(\gamma)$ in the expression of $P$. Given an element $\lambda = (\lambda_\gamma)_{\gamma\in G^*}$ in $\mathcal{P}_G$, we set

$$s_\lambda = \prod_{\gamma\in G^*} s_{\lambda_\gamma}(\gamma).$$

The set of complex irreducible characters of $G_n$ is a basis of the algebra of complex-valued class functions of $G_n$, so that we can denote this latter by $\mathbb{C}\,\mathrm{Irr}(G_n)$. The direct sum

$$R(G) = \bigoplus_{n\geq 0} \mathbb{C}\,\mathrm{Irr}(G_n)$$

can then be endowed with the structure of a commutative and cocommutative $\mathbb{N}$-graded Hopf algebra, where the product is given by (the maps induced on the level of characters by) the induction functors $\mathrm{Ind}_{G_m\times G_n}^{G_{m+n}}$ and the coproduct is afforded likewise by the restriction functors $\mathrm{Res}_{G_m\times G_n}^{G_{m+n}}$ [14, I, Appendix B, 4 and I, 7, Example 26]. Since $\Lambda_{\mathbb{C}}(G)$ is a free commutative algebra, there is a unique homomorphism of $\mathbb{C}$-algebras

$$\mathrm{ch}^{-1} : \Lambda_{\mathbb{C}}(G) \to R(G)$$

with the following property: for each $n \geq 0$ and each $c \in G_*$, $\mathrm{ch}^{-1}$ maps the variable $p_n(c)$ to the characteristic function of the conjugacy class of $G_n$ consisting of the products $(g_1, \dots, g_n)\,\sigma$, where the permutation $\sigma \in \mathfrak{S}_n$ is a $n$-cycle and the product $g_1g_2\cdots g_n$ belongs to the conjugacy class $c$. It turns out that $\mathrm{ch}^{-1}$ is an isomorphism of Hopf algebra, whose inverse will be denoted by ch. Then, using arguments of orthogonality and integrality, it can be shown [14, I, Appendix B, 9] that the image under ch of the irreducible characters of $G_n$ are the elements $s_\lambda$, where $\lambda \in \mathcal{P}_G$ is such that $|\lambda| = n$.

Later on, we will need to know the image under ch of characters $\eta_n(\rho)$. We do the computation now.

**Lemma 6.1.** *Let $\gamma_1$, …, $\gamma_s$ the irreducible characters of $G$, let $c_1$, …, $c_s$ be non-negative integers, and set $\rho = c_1\gamma_1 + \cdots + c_s\gamma_s$. Then*

$$\sum_{n\geq 0} \mathrm{ch}\big(\eta_n(\rho)\big) = \prod_{i=1}^{s} \left(\sum_{n\geq 0} h_n(\gamma_i)\right)^{c_i}.$$

*Proof.* The proof given in [14, I, Appendix B, 8] for the case where $\rho$ is irreducible can be easily adapted. Indeed in the computation that follows Equation (8.2) in that reference, the steps that lead to the equality

$$\sum_{n\geq 0} \mathrm{ch}\big(\eta_n(\gamma)\big) = \exp\left(\sum_{r\geq 1} \frac{1}{r} \sum_{c\in G_*} \zeta_c^{-1}\gamma(c)p_r(c)\right)$$

are valid even if the character $\gamma$ is reducible. Applying this formula to the character $\rho$, we get

$$\begin{aligned}\sum_{n\geq 0}\mathrm{ch}\big(\eta_n(\rho)\big) &= \exp\left(\sum_{r\geq 1}\frac{1}{r}\sum_{c\in G_*}\zeta_c^{-1}\rho(c)p_r(c)\right)\\ &= \prod_{i=1}^{s}\left[\exp\left(\sum_{r\geq 1}\frac{1}{r}\sum_{c\in G_*}\zeta_c^{-1}\gamma_i(c)p_r(c)\right)\right]^{c_i}\\ &= \prod_{i=1}^{s}\left[\exp\left(\sum_{r\geq 1}\frac{1}{r}p_r(\gamma_i)\right)\right]^{c_i}\\ &= \prod_{i=1}^{s}\Big[\sum_{n\geq 0}h_n(\gamma_i)\Big]^{c_i},\end{aligned}$$

the last step in the computation coming from Newton's formulas. $\square$

**The comparison result.** Having now recalled Specht's construction of the irreducible characters for the wreath product $G \wr \mathfrak{S}_n$ of an arbitrary finite group $G$ by the symmetric group $\mathfrak{S}_n$, we can specialize to the case where $G$ is the group $W = \mathbb{Z}/2\mathbb{Z}$ with two elements. The notation $W_n$ for the wreath product $W \wr \mathfrak{S}_n$ then agrees with its use by Bonnafé and Hohlweg. The Hopf algebra $R(W)$ is identical to the complexified Hopf algebra $\mathcal{CHAR}\otimes_{\mathbb{Z}}\mathbb{C}$. The set $W^*$ of irreducible characters of $W$ has two elements, namely the trivial character $\tau$ and the signature $\varepsilon$. One can view an element $\lambda = (\lambda_\tau, \lambda_\varepsilon)$ of $\mathcal{P}_W$ as a bipartition $(\lambda^+, \lambda^-)$ by setting $\lambda^+ = \lambda_\tau$ and $\lambda^- = \lambda_\varepsilon$. As a final piece of notation, we set $\lambda^* = \big(\lambda^+, (\lambda^-)^t\big)$ for any bipartition $\lambda = (\lambda^+, \lambda^-)$.

Generalizing Poirier and Reutenauer's work [17] for symmetric groups to the case of $W_n$, we define a linear map:

$$f:\ \mathcal{Q}\otimes_{\mathbb{Z}}\mathbb{C} \quad\longrightarrow\quad \Lambda_{\mathbb{C}}(W)$$

by setting $f(z_Q) = s_{(\mathrm{sh}\,Q)^*}$ for any bitableau $Q$. With all these notations, our result can be stated as follows:

**Theorem 6.2.** *The following diagram of* $\mathbb{N}$*-graded Hopf algebras*

$$\begin{tikzcd} \mathcal{Q}\otimes_{\mathbb{Z}}\mathbb{C} \arrow[r, two heads, "f"] \arrow[dr, "\widetilde{\theta}"] & \Lambda_{\mathbb{C}}(W) \\ \Sigma'\otimes_{\mathbb{Z}}\mathbb{C} \arrow[u, hook, "i"] \arrow[r, two heads, "\theta"'] & \mathcal{CHAR}\otimes_{\mathbb{Z}}\mathbb{C} \arrow[u, "\sim"', "\mathrm{ch}"] \end{tikzcd}$$

*is commutative. In particular* $\mathrm{ch}(\xi_\lambda) = s_{\lambda^*}$*, for any bipartition* $\lambda$*, so that Bonnafé and Hohlweg's construction is equivalent to Specht's one, up to a relabelling.*

Some further notation and a bijection will be needed for the proof. We present them now.

**Some notations and a bijection.** We call quasicomposition a sequence $E = (e_1, e_2, e_3, \ldots)$ of non-negative integers, all of whose terms but a finite number vanish. The size $|E|$ of $E$ is the sum $e_1 + e_2 + e_3 + \cdots$ of the terms. Given a partition $\mu$ and a quasicomposition $E$, we denote by $\mathrm{Tab}\,(\mu, E)$ the set of all semistandard tableau of shape $\mu$ and weight $E$, that is the set of all fillings of the Ferrers diagram of shape $\mu$ with positive integers, in such a way that the numbers are weakly increasing from left to right in the rows, strictly increasing from top to bottom in the columns, and that there is $e_1$ times the number 1, $e_2$ times the number 2, and so on [14, p. 5]. The set $\mathrm{Tab}\,(\mu, E)$ is of course empty unless $|\mu| = |E|$. Given any quasicomposition $E = (e_1, e_2, e_3, \ldots)$, the formula

$$h_{e_1} h_{e_2} h_{e_3} \cdots = \sum_{\mu \in \mathcal{P}} \bigl|\mathrm{Tab}\,(\mu, E)\bigr| s_\mu$$

holds in $\Lambda_{\mathbb{C}}$ (see [14, I, (6.4)] for a proof).

Now we fix a positive integer $n$ and a signed composition $C = (c_1, \ldots, c_\ell)$ of it. Let $\ell$ be the length of $C$. We define $\mathrm{Comp}(C)$ as the set of all quasicompositions $D = (d_1, \ldots, d_\ell)$ such that $d_i = 0$ if $c_i > 0$ and $0 \le d_i \le -c_i$ if $c_i < 0$. Given such a $D$, we further define two quasicompositions $T_{C,D} = (t_1, \ldots, t_\ell)$ and $E_{C,D} = (e_1, \ldots, e_\ell)$ by

$$t_i = \begin{cases} c_i & \text{if } c_i > 0, \\ d_i & \text{if } c_i < 0; \end{cases} \quad \text{and} \quad e_i = \begin{cases} 0 & \text{if } c_i > 0, \\ -c_i - d_i & \text{if } c_i < 0. \end{cases}$$

The signed composition obtained by omitting the zeros in the list

$$(-e_1, t_1, -e_2, t_2, \ldots, -e_\ell, t_\ell)$$

will be denoted by $B_{C,D}$. For instance, for $C = (2, \bar{2}, \bar{3}, 1, \bar{1}, 2, 2, \bar{2}) \,||\!\!\models 15$, we can choose $D = (0, 0, 2, 0, 1, 0, 0, 0)$, and then $T_{C,D} = (2, 0, 2, 1, 1, 2, 2, 0)$, $E_{C,D} = (0, 2, 1, 0, 0, 0, 0, 2)$ and $B_{C,D} = (2, \bar{2}, \bar{1}, 2, 1, 1, 2, 2, \bar{2})$.

Finally, given a bipartition $\lambda = (\lambda^+, \lambda^-)$ and a signed composition $C$ with $|\lambda| = |C|$, we define $\mathrm{Bitab}(\lambda, C)$ as the set of all standard bitableaux $Q$ such that $\mathrm{sh}(Q) = \lambda^*$ and $C \leftarrow \mathbf{C}(Q)$ (see Remark 4.7).

One of the key to the proof of Theorem 6.2 is the following combinatorial result.

**Proposition 6.3.** *Given a bipartition $\lambda$ and a signed composition $C$ with $|\lambda| = |C|$, the sets* $\mathrm{Bitab}(\lambda, C)$ *and*

$$\coprod_{D \in \mathrm{Comp}(C)} \mathrm{Tab}\left(\lambda^+, T_{C,D}\right) \times \mathrm{Tab}\left(\lambda^-, E_{C,D}\right)$$

*have the same cardinality.*

*Proof.* Let $n$ be a positive integer, $C$ be a signed composition of $n$, and $\lambda = (\lambda^+, \lambda^-)$ be a bipartition with $|\lambda| = n$. We construct inverse bijections between $\mathrm{Bitab}(\lambda, C)$ and

$$\coprod_{D \in \mathrm{Comp}(C)} \mathrm{Tab}\left(\lambda^+, T_{C,D}\right) \times \mathrm{Tab}\left(\lambda^-, E_{C,D}\right)$$

as follows.

First let $(R, S)$ be in the second set, so that $R \in \mathrm{Tab}\,(\lambda^+, T_{C,D})$ and $S \in \mathrm{Tab}\,(\lambda^-, E_{C,D})$ for some $D \in \mathrm{Comp}(C)$. We can put a total order on the boxes in $R$ and $S$ by requiring that:

- A box is smaller than another one if the label written in it is smaller than the one in the other.
- Given two boxes with the same label in it, a box in $S$ is smaller than a box in $R$.
- For boxes containing the same label and located in the same tableau ($R$ or $S$), boxes located south-west are smaller than boxes located north-east.

We can then enumerate in increasing order the boxes in $R$ and $S$. Filling now each box of $R$ and $S$ by its rank of appearance in the enumeration, we construct a standard bitableau $\tilde{Q}$ of shape $\lambda$. We then define $Q$ as the bitableau obtained from $\tilde{Q}$ by transposing $\tilde{Q}^-$, so that $Q$ has shape $\lambda^*$. Comparing this construction with the combinatorial rule in Remark 4.7 that computes $\boldsymbol{C}(Q)$, we easily check that the signed composition $B_{C,D}$ can be obtained from $\mathbf{C}(Q)$ by refinement of the parts, so that $C \xleftarrow{B} B_{C,D} \xleftarrow{R} \mathbf{C}(Q)$, which implies $Q \in \mathrm{Bitab}(\lambda, C)$.

In the other direction, let $Q$ be a given element in $\mathrm{Bitab}(\lambda, C)$. From Theorem 3.15, there exists a unique signed composition $B$ such that $C \xleftarrow{B} B \xleftarrow{R} \mathbf{C}(Q)$, and we can find a (unique) element $D \in \mathrm{Comp}(C)$ so that $B = B_{C,D}$. Now we transpose $Q^-$ and get a bitableau $\tilde{Q}$. We construct a list $L = (l_1, l_2, \ldots, l_n)$ of positive integers by placing first $|c_1|$ times the number 1, then $|c_2|$ times the number 2, and so on. Then we substitute $l_1$ to 1, $l_2$ to 2, and so on, in the boxes of $\tilde{Q}$, and obtain in this way a pair $(R, S)$ of tableaux of shapes $\lambda^+$ and $\lambda^-$ respectively. The fact that $B_{C,D} \xleftarrow{R} \mathbf{C}(Q)$ implies that this construction yield two semistandard tableaux $R$ and $S$ with weights $T_{C,D}$ and $E_{C,D}$ respectively, that is to say

$$(R, S) \in \mathrm{Tab}\left(\lambda^+, T_{C,D}\right) \times \mathrm{Tab}\left(\lambda^-, E_{C,D}\right).$$

It is a routine task to check that the two above constructions yield mutually inverse bijections between $\coprod_{D \in \mathrm{Comp}(C)} \mathrm{Tab}\left(\lambda^+, T_{C,D}\right) \times \mathrm{Tab}\left(\lambda^-, E_{C,D}\right)$ and $\mathrm{Bitab}(\lambda, C)$. $\square$

We end this paragraph by an example that illustrates the constructions needed in the proof above. We take $n = 15$ and choose the same signed composition $C$ as in the previous example, namely

$$C = (2, \bar{2}, \bar{3}, 1, \bar{1}, 2, 2, \bar{2}).$$

We choose $\lambda^+ = 631$ and $\lambda^- = 41$, so that $\lambda^* = (631, 21^3)$. Starting from the pair $(R, S)$ with

$$R = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 3 & 3 & 4 & 7 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 6 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad S = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 8 \\ \hline 8 \\ \cline{1-1}\end{array},$$

we construct $\tilde{Q} = (\tilde{Q}^+, \tilde{Q}^-)$ where

$$\tilde{Q}^+ = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 6 & 7 & 8 & 13 \\ \hline 9 & 11 & 12 \\ \cline{1-3} 10 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \tilde{Q}^- = \begin{array}{|c|c|c|c|}\hline 3 & 4 & 5 & 15 \\ \hline 14 \\ \cline{1-1}\end{array},$$

whence $Q = (Q^+, Q^-)$ with

$$Q^+ = \tilde{Q}^+ = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 6 & 7 & 8 & 13 \\ \hline 9 & 11 & 12 \\ \cline{1-3} 10 \\ \cline{1-1}\end{array} \qquad \text{and} \qquad Q^- = {}^t\tilde{Q}^- = \begin{array}{|c|c|}\hline 3 & 14 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 15 \\ \cline{1-1}\end{array}.$$

Since $\mathbf{C}(Q) = (2, \bar{3}, 3, 1, 4, \bar{2})$, it holds that $C \xleftarrow{B} B \xleftarrow{R} \mathbf{C}(Q)$ with

$$B = (2, \bar{2}, \bar{1}, 2, 1, 1, 2, 2, \bar{2}),$$

which implies $C \leftarrow \mathbf{C}(Q)$.

In the other direction, we start from the bitableau $Q$. We observe that the signed composition $B$ such that $C \xleftarrow{B} B \xleftarrow{R} \mathbf{C}(Q)$ is $B_{C,D}$, where $D$ is given by $D = (0, 0, 2, 0, 1, 0, 0, 0)$. Now we write down the list

$$L = (1, 1, 2, 2, 3, 3, 3, 4, 5, 6, 6, 7, 7, 8, 8)$$

from $C$. Transposing the negative tableau $Q^-$, we write down $\tilde{Q}$ and substitute the elements of $L$ to the numbers in the boxes of $\tilde{Q}$. We recover our original pair $(R, S)$. We easily verify that $R$ has weight

$$T_{C,D} = (2, 0, 2, 1, 1, 2, 2, 0)$$

and that $S$ has weight

$$E_{C,D} = (0, 2, 1, 0, 0, 0, 0, 2).$$

**Proof of Theorem 6.2.**

**1** We first compute the image by ch of the induced character $\mathrm{Ind}_{\mathfrak{S}_n}^{W_n} 1$ of $W_n$, where $n$ is a positive integer. To do that, we construct the complex representation $\eta_n(V)$ of $W_n$, where $V$ is the left regular representation of $W = \mathbb{Z}/2\mathbb{Z}$. Denoting by $\mathbb{C}_1$ the trivial representation of $\mathfrak{S}_n$, we then observe that the isomorphism of vector spaces $\mathrm{Ind}_{\mathfrak{S}_n}^{W_n} \mathbb{C}_1 \cong \eta_n(V)$ given by the sequence of natural identifications

$$\mathrm{Ind}_{\mathfrak{S}_n}^{W_n} \mathbb{C}_1 \cong \mathbb{C}W_n \otimes_{\mathbb{C}\mathfrak{S}_n} \mathbb{C}_1 \cong \mathbb{C}(W^n) \cong (\mathbb{C}W)^{\otimes n} = V^{\otimes n} = \eta_n(V)$$

is compatible with the action of $W_n$. Since $V$ has $\tau + \varepsilon$ for character, it follows that $\mathrm{Ind}_{\mathfrak{S}_n}^{W_n} 1 = \eta_n(\tau + \varepsilon)$. Lemma 6.1 now implies that

$$\sum_{n \geq 0} \mathrm{ch}\big(\mathrm{Ind}_{\mathfrak{S}_n}^{W_n} 1\big) = \left(\sum_{n \geq 0} h_n(\tau)\right)\left(\sum_{n \geq 0} h_n(\varepsilon)\right).$$

On the other side, it is easy to check that $\eta_n(\tau)$ is the trivial character of $W_n$. Therefore ch maps the trivial character $\mathrm{Ind}_{W_n}^{W_n} 1$ of $W_n$ to $h_n(\tau)$. To comply with the philosophy used by Bonnafé and Hohlweg, we will write for any positive integer $n$

$$\varphi_{\pm n} = \mathrm{ch}\big(\mathrm{Ind}_{W_{\pm n}}^{W_n} 1\big) = \begin{cases} \mathrm{ch}\big(\mathrm{Ind}_{W_n}^{W_n} 1\big) = h_n(\tau) & \text{for '+' sign,} \\ \mathrm{ch}\big(\mathrm{Ind}_{\mathfrak{S}_n}^{W_n} 1\big) = \sum_{k=0}^{n} h_k(\tau) h_{n-k}(\varepsilon) & \text{for '−' sign.} \end{cases}$$

**2** We now prove the equality $f \circ i = \mathrm{ch} \circ \theta$. Given any signed composition $C = (c_1, \ldots, c_\ell)$, there holds $x_C = x_{c_1} \cdots x_{c_\ell}$. Since $\theta$ is a morphism of Hopf algebras, we can write

$$\mathrm{Ind}_{W_C}^{W_{|C|}} 1_C = \theta(x_C) = \theta(x_{c_1}) \cdots \theta(x_{c_\ell}) = \mathrm{Ind}_{W_{c_1}}^{W_{|c_1|}} 1 \bullet \cdots \bullet \mathrm{Ind}_{W_{c_\ell}}^{W_{|c_\ell|}} 1,$$

and taking its image under ch,

$$\mathrm{ch}\,\mathrm{Ind}_{W_C}^{W_{|C|}} = \mathrm{ch} \circ \theta(x_C) = \varphi_{c_1} \cdots \varphi_{c_\ell}.$$

The formula

$$\varphi_{-n} = \sum_{k=0}^{n} h_k(\tau) h_{n-k}(\varepsilon),$$

valid for any positive integer $n$, makes possible to continue the computation:

$$\mathrm{ch} \circ \theta(x_C) = \sum_{D \in \mathrm{Comp}(C)} h_{t_1}(\tau) \cdots h_{t_\ell}(\tau)\ h_{e_1}(\varepsilon) \cdots h_{e_\ell}(\varepsilon),$$

where the quasicompositions $(t_1, \ldots, t_\ell)$ and $(e_1, \ldots, e_\ell)$ appearing in the sum are $T_{C,D}$ and $E_{C,D}$ respectively. We thus get, using Proposition 6.3 and the decomposition of $X_C$ given at the end of Remark 4.7:

$$\begin{aligned}
\mathrm{ch} \circ \theta(x_C) &= \sum_{D \in \mathrm{Comp}(C)} \left( \sum_{\lambda^+ \in \mathcal{P}} \left| \mathrm{Tab}\left(\lambda^+, T_{C,D}\right) \right| s_{\lambda^+}(\tau) \right) \left( \sum_{\lambda^- \in \mathcal{P}} \left| \mathrm{Tab}\left(\lambda^-, E_{C,D}\right) \right| s_{\lambda^-}(\varepsilon) \right) \\
&= \sum_{(\lambda^+, \lambda^-) \in \mathcal{P}_W} \left( \sum_{D \in \mathrm{Comp}(C)} \left| \mathrm{Tab}\left(\lambda^+, T_{C,D}\right) \times \mathrm{Tab}\left(\lambda^-, E_{C,D}\right) \right| \right) s_{\lambda^+}(\tau)\ s_{\lambda^-}(\varepsilon) \\
&= \sum_{\lambda \in \mathcal{P}_W} \left| \mathrm{Bitab}(\lambda, C) \right| s_\lambda \\
&= \sum_{\substack{Q \text{ std. bitableau} \\ C \leftarrow \boldsymbol{C}(Q)}} s_{(\mathrm{sh}\, Q)^*} \\
&= \sum_{\substack{Q \text{ std. bitableau} \\ C \leftarrow \boldsymbol{C}(Q)}} f(z_Q) \\
&= f \circ i(x_C).
\end{aligned}$$

Since the elements $x_C$ generate $\Sigma' \otimes_{\mathbb{Z}} \mathbb{C}$ as a vector space, it follows that $\mathrm{ch} \circ \theta = f \circ i$.

**3** To complete the proof, it now suffices to show that $f = \mathrm{ch} \circ \tilde{\theta}$. We first observe that both members of this equality coincide on the image of $i$ in $\mathcal{Q} \otimes_{\mathbb{Z}} \mathbb{C}$, since

$$\mathrm{ch} \circ \theta = f \circ i \qquad \text{and} \qquad \theta = \tilde{\theta} \circ i.$$

On the other hand, $f$ and $\mathrm{ch} \circ \tilde{\theta}$ have the same kernel, namely the vector space $\mathcal{Q}_n^\perp$ spanned by the elements $z_Q - z_{Q'}$ for standard bitableaux $Q$ and $Q'$ of the same shape (see Theorem 4.14). Since $\theta$ is surjective, this kernel, together with the image of $i$, spans $\mathcal{Q} \otimes_{\mathbb{Z}} \mathbb{C}$. The result follows easily from these facts.